

Convergence of a decoupled mixed FEM for miscible displacement in interfacial porous media

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Abstract

In this paper, we study the stability and convergence of a decoupled and linearized mixed finite element method (FEM) for incompressible miscible displacement in a porous media whose permeability and porosity are discontinuous across some interfaces. We show that the proposed scheme has optimal-order convergence rate unconditionally, without restriction on the grid ratio (between the time-step size and spatial mesh size). Previous works all required certain restrictions on the grid ratio except for the problem with globally smooth permeability and porosity. Our idea is to introduce an intermediate system of elliptic interface problems, whose solution is uniformly regular in each subdomain separated by the interfaces and its finite element solution coincides with the fully discrete solution of the original problem. In order to prove the boundedness of the fully discrete solution, we study the finite element discretization of the intermediate system of elliptic interface problems.

1 Introduction

Numerical computation of miscible displacement in porous media has attracted much attention in recent decades due to its applications in reservoir simulations and exploration of underground oil; see [2, 8, 10, 12, 14, 27]. The model describes the motion of a miscible fluid of two (or more) components in porous media, where the velocity of the fluid is given by Darcy's law

$$\mathbf{u} = -\frac{k(x)}{\mu(c)} \nabla p.$$

In the last equation, p denotes the pressure of the fluid mixture, $k(x)$ denotes the permeability of the porous media, and $\mu(c)$ is the viscosity of the fluid depending on the concentration c of the first component. The incompressibility of the fluid is described by

$$\nabla \cdot \mathbf{u} = q_I - q_P,$$

where q_I and q_P are given injection and production sources. The concentration c is governed by a convection-diffusion equation

$$\Phi(x) \frac{\partial c}{\partial t} - \nabla \cdot (D(\mathbf{u}, x) \nabla c) + \mathbf{u} \cdot \nabla c = \hat{c} q_I - c q_I.$$

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where $\Phi(x)$ denotes the porosity of the media and $D(\mathbf{u}, x)$ denotes the diffusion-dispersion tensor, which is given by [5, 6]

$$D(\mathbf{u}, x) = \Phi(x) \left[d_0 I + F(\text{Pe}) |\mathbf{u}| \left(\alpha_1 I + (\alpha_2 - \alpha_1) \frac{\mathbf{u} \otimes \mathbf{u}}{|\mathbf{u}|^2} \right) \right].$$

In this formula, $F(\text{Pe}) = \text{Pe}/(\text{Pe} + d_r)$ is a function of the local molecular Peclet number $\text{Pe} = d_p |\mathbf{u}|$, where d_0 , α_1 , α_2 , d_r and d_p are positive constants related to the porous media. It is straightforward to verify that

$$d_1 |\xi|^2 \leq D(\mathbf{u}, x) \xi \cdot \xi \leq (d_2 + d_3 |\mathbf{u}|) |\xi|^2, \quad \forall \xi \in \mathbb{R}^d,$$

for some positive constants d_1 , d_2 and d_3 .

Existence of weak or semiclassical solutions for the miscible displacement equations was studied in [8, 20], and numerical analysis of the model has been done by many authors. In particular, a Galerkin FEM was studied by Ewing and Wheeler [19], and a Galerkin-mixed FEM was analyzed by Douglas et al [13], where the Galerkin method was used to solve the parabolic concentration equation and a mixed FEM was applied to solve the elliptic pressure equation. For both methods, a linearized semi-implicit Euler scheme was used for the time stepping and optimal error estimates were presented roughly under the restriction $\tau = o(h)$. In [15], a characteristic method was applied to the parabolic concentration equation and the mixed FEM was used to solve the elliptic pressure equation. Optimal error estimates were established under the same condition, i.e. $\tau = o(h)$. More recently, a Galerkin method combined with a post-process technique was studied in [26], an Euler–Lagrange localized approximation method was studied in [34] and a modified method of characteristics combined with mixed FEM was studied in [32]. In all these works, error estimates were derived with certain restrictions on the grid ratio. To remove these restrictive conditions, a new approach was introduced in [23, 24] to decouple the discretization errors from the temporal and spatial directions, and optimal error estimates of a Galerkin-mixed FEM was established without restriction on the grid ratio. The methodology of [23, 24] was later successfully applied to other nonlinear parabolic equations, such as the nonlinear Schrödinger equation [35], the thermistor equations [17] and the Navier-Stokes equations [31]. However, all the analyses presented in these works rely on the global H^2 regularity of the “time-discrete solution” (the solution of the linearized PDEs), which requires the permeability and porosity to be globally smooth in the miscible displacement model.

In engineering computations, due to the existence of fault, filling-type karst caves or complex geological composition, the permeability and porosity are often discontinuous across some interfaces. It is desirable to solve the miscible displacement equations with discontinuous permeability and porosity by stable and accurate numerical methods. For this purpose, numerical methods for flow in porous media with discontinuous permeability have been studied by many authors based on linear models. For example, see [3, 7, 18, 29] on the approximation of the elliptic pressure equation and see [10] on the approximation of a parabolic pressure equation from the compressible model. Convergence of finite element methods for general linear elliptic and parabolic interface problems can also be found in [9, 25, 30].

In this paper, we study stability and convergence of fully discrete FEMs for the full model of miscible displacement in porous media, where the permeability and porosity are discontinuous across some interfaces. Mathematically, we assume that the domain Ω is partitioned into $\Omega = \cup_{m=0}^M \Omega_m$ separated by the interfaces Γ_m , $m = 1, \dots, M$, as shown in Figure 1.1, and we

consider the nonlinear elliptic-parabolic interface problem

$$\left\{ \begin{array}{ll} \Phi(x) \frac{\partial c}{\partial t} - \nabla \cdot (D(\mathbf{u}, x) \nabla c) + \mathbf{u} \cdot \nabla c = \hat{c}q_I - cq_I & \text{in } \Omega_m, \\ [c] = 0, \quad [D(\mathbf{u}, x) \nabla c \cdot \mathbf{n}] = 0 & \text{on } \Gamma_m, \\ D(\mathbf{u}, x) \nabla c \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \\ c(x, 0) = c_0(x) & \text{for } x \in \Omega, \end{array} \right. \quad (1.1)$$

$$\left\{ \begin{array}{ll} \nabla \cdot \mathbf{u} = q_I - q_P & \text{in } \Omega_m, \\ \mathbf{u} = -\frac{k(x)}{\mu(c)} \nabla p & \text{in } \Omega_m, \\ [p] = 0, \quad [\mathbf{u} \cdot \mathbf{n}] = 0 & \text{on } \Gamma_m, \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{array} \right. \quad (1.2)$$

In each subdomain Ω_m , the pressure p , the velocity \mathbf{u} and the concentration c are governed by the partial differential equations, and jump conditions are specified across the interfaces. The permeability $k(x)$ and porosity $\Phi(x)$ are assumed to be constant in each subdomain Ω_m but are discontinuous across the interfaces Γ_m .

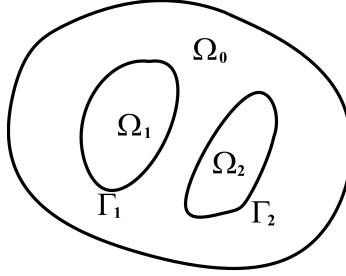


Figure 1.1: The domain and the interfaces.

Clearly, the diffusion-dispersion tensor $D(\mathbf{u}, x)$ is an unbounded function of \mathbf{u} . Due to this strong nonlinearity and the coupling of equations, previous error estimates presented for the linear interface problems cannot be extended here. A direct application of the traditional error estimates requires undesired restrictions on the grid ratio to control the numerical velocity. In order to avoid any restrictive conditions on the grid ratio, one has to use the error-splitting technique introduced in [23, 24]. However, due to the discontinuity of the permeability and porosity across the interfaces, the solution of (1.1)-(1.2) is not globally smooth. Instead, they are at most piecewise smooth [4], as assumed in this paper. In this case, the analysis for the Galerkin-mixed FEM presented in [24] does not work. In this paper, we show that a decoupled and linearized mixed FEM is stable for the nonlinear interface problem by proving that the time-discrete solution is piecewise smooth enough in each subdomain separated by the interfaces. Optimal error estimates are established without restriction on the grid ratio. We believe that the methodology of this paper, together with Lemma 2.1-2.2 introduced here, can also be applied to other nonlinear parabolic interface problems in engineering and physics.

2 Main results

Suppose that the smooth domain Ω is partitioned into $\Omega = \cup_{m=0}^M \Omega_m$, where Ω_m is enclosed by a smooth interface Γ_m for $m = 1, \dots, M$, and $\Gamma_0 = \partial\Omega$. For any integer $s \geq 0$ and a subdomain Ω_m , we let $W^{s,p}(\Omega_m)$ and $H^s(\Omega_m) := W^{s,2}(\Omega_m)$ denote the usual Sobolev spaces of functions defined on the domain Ω_m ; see [1]. Let L^p denote the abbreviations of $L^p(\Omega)$ and define $\overline{W}^{k,p}$ as the subspace of L^p equipped with the norm

$$\|f\|_{\overline{W}^{s,p}} := \sum_{m=0}^M \|f\|_{W^{s,p}(\Omega_m)}.$$

Therefore, the functions in $\overline{W}^{s,p}$ are in $W^{s,p}(\Omega_m)$ for each subdomain Ω_m , but may not be continuous in the whole domain Ω . To simplify the notations, we define $\overline{H}^s := \overline{W}^{s,2}$, $\overline{L}^p := \overline{W}^{0,p}$ and

$$(f, g) = \sum_{m=0}^M \int_{\Omega_m} f(x)g(x) \, dx, \quad \text{for } f, g \in \overline{L}^2.$$

For any Banach space X and a function $g : (0, T) \rightarrow X$, we define the norm

$$\|g\|_{L^p((0,T);X)} = \begin{cases} \left(\int_0^T \|g(t)\|_X^p \, dt \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \text{ess sup}_{t \in (0,T)} \|g(t)\|_X, & p = \infty. \end{cases}$$

Let $\{t_n\}_{n=0}^N$ be a uniform partition of the time interval $[0, T]$ with $\tau = T/N$ and denote

$$p^n(x) = p(x, t_n), \quad \mathbf{u}^n(x) = \mathbf{u}(x, t_n), \quad c^n(x) = c(x, t_n).$$

For any sequence of functions $\{f^n\}_{n=0}^N$, we define $D_\tau f^{n+1} = (f^{n+1} - f^n)/\tau$. Let π_h denote a quasi-uniform partition of $\cup_{m=0}^M \Omega_m$ into triangles (or tetrahedrons) T_j , $j = 1, \dots, J$. For a triangle T_j with two vertices on the boundary $\partial\Omega$ or an interface Γ_m , we define \tilde{T}_j to be a triangle with one curved side which fit the boundary or the interfaces exactly, with the same vertices as T_j . Let $h = \max_{1 \leq j \leq J} \{\text{diam } T_j\}$ denote the mesh size, and let \mathbb{P}_r denote the space of polynomials of degree $r \geq 1$. We define the discontinuous finite element space

$$S_h^r = \{w_h \in L^2(\Omega) : w_h|_{T_j} \in \mathbb{P}_r \text{ for each element } T_j \in \pi_h \text{ and } \int_\Omega w_h \, dx = 0\}.$$

Let $S_h^r(\Gamma_m)$ denote the space of functions in $S_h^r(\Omega_m)$ restricted to Γ_m . To simplify the notations, we define $\Gamma_0 = \partial\Omega$, $\Gamma = \bigcup_{m=0}^M \Gamma_m$, and define $S_h^r(\Gamma)$ as the space of functions on Γ whose restriction to Γ_m is in $S_h^r(\Gamma_m)$. Let \mathbf{H}_Γ^1 be the space of vector-valued functions $\mathbf{v} \in (\overline{H}^1)^d$ such that $\nabla \cdot \mathbf{v} \in L^2$, $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $[\mathbf{v} \cdot \mathbf{n}] = 0$ on Γ_m , $m = 1, \dots, M$. Let \mathbf{H}_h^r denote the Raviart–Thomas mixed finite element subspace of \mathbf{H}_Γ^1 introduced in [16, 28, 33], which coincides with an element of $\mathbb{P}_r^d \oplus \mathbf{x}\mathbb{P}_r$ in each triangle T_j . Moreover, we require that the functions $\mathbf{v}_h \in \mathbf{H}_h^r$ satisfy the boundary condition $\int_{\tilde{e}_j} \mathbf{v}_h \cdot \mathbf{n} \chi_h \, ds = 0$, $\forall \chi_h \in S_h^r$, on each boundary edge \tilde{e}_j and the jump condition $\int_{\tilde{e}_j} [\mathbf{v}_h \cdot \mathbf{n}] \chi_h \, ds = 0$, $\forall \chi_h \in S_h^r$, on each interface edge \tilde{e}_j . Then we have $\nabla \cdot \mathbf{v}_h \in S_h^r$ for $\mathbf{v}_h \in \mathbf{H}_h^r$.

To approximate p , c , \mathbf{u} and $\mathbf{w} = -D(\mathbf{u}, x)\nabla c$, we look for $P_h^n, \mathcal{C}_h^n \in S_h^r$ and $\mathbf{U}_h^n, \mathbf{W}_h^n \in \mathbf{H}_h^r$ which satisfy the equations

$$\left(\frac{\mu(\mathcal{C}_h^n)}{k(x)} \mathbf{U}_h^n, \mathbf{v}_h \right) = \left(P_h^n, \nabla \cdot \mathbf{v}_h \right), \quad (2.1)$$

$$\left(\nabla \cdot \mathbf{U}_h^n, \varphi_h \right) = \left(q_I^n - q_P^n, \varphi_h \right), \quad (2.2)$$

$$\left(D(\mathbf{U}_h^n, x)^{-1} \mathbf{W}_h^{n+1}, \bar{\mathbf{v}}_h \right) = \left(\mathcal{C}_h^{n+1}, \nabla \cdot \bar{\mathbf{v}}_h \right), \quad (2.3)$$

$$\begin{aligned} \left(\Phi(x) D_\tau \mathcal{C}_h^{n+1}, \bar{\varphi}_h \right) + \left(\nabla \cdot \mathbf{W}_h^{n+1}, \bar{\varphi}_h \right) - \left(D(\mathbf{U}_h^n, x)^{-1} \mathbf{U}_h^n \cdot \mathbf{W}_h^{n+1}, \bar{\varphi}_h \right) \\ = \left(\hat{c}^{n+1} q_I^{n+1} - \mathcal{C}_h^{n+1} q_I^{n+1}, \bar{\varphi}_h \right), \end{aligned} \quad (2.4)$$

for any $\mathbf{v}_h, \bar{\mathbf{v}}_h \in \mathbf{H}_h^r$ and $\varphi_h, \bar{\varphi}_h \in S_h^r$, where $n = 0, 1, 2, \dots$, and the initial data \mathcal{C}_h^0 is chosen as the Lagrangian interpolation of c^0 .

For the initial-boundary value problem (1.1)-(1.2) to be well-posed, we require the compatibility condition

$$\int_{\Omega} q_I dx = \int_{\Omega} q_P dx, \quad (2.5)$$

and the physical restrictions

$$\|q_I\|_{L^\infty} + \|q_P\|_{L^\infty} \leq q_0, \quad (2.6)$$

$$k_0^{-1} \leq k(x) \leq k_0 \quad \text{for } x \in \Omega, \quad (2.7)$$

$$\Phi_0^{-1} \leq \Phi(x) \leq \Phi_0 \quad \text{for } x \in \Omega, \quad (2.8)$$

$$\mu_0^{-1} \leq \mu(s) \leq \mu_0 \quad \text{and} \quad |\mu'(s)| \leq \mu_0 \quad \text{for } s \in \mathbb{R}, \quad (2.9)$$

for some positive constants q_0 , k_0 , Φ_0 and μ_0 . Moreover, we assume that the solution of the initial-boundary value problem (1.1)-(1.2) exists and possesses certain piecewise regularity such as

$$\begin{aligned} \|p\|_{L^\infty((0,T);\bar{H}^{r+2})} + \|\mathbf{u}\|_{L^\infty((0,T);\bar{H}^{r+1})} + \|\partial_t \mathbf{u}\|_{L^\infty((0,T);\bar{H}^1)} + \|\partial_{tt} \mathbf{u}\|_{L^2((0,T);\bar{L}^2)} \\ + \|c\|_{L^\infty((0,T);\bar{H}^{r+2})} + \|\partial_t c\|_{L^\infty((0,T);\bar{H}^{r+1})} + \|\partial_{tt} c\|_{L^2((0,T);\bar{L}^2)} \leq C_0, \end{aligned} \quad (2.10)$$

for some positive constant C_0 .

The main result of this paper is the following theorem.

Theorem 2.1 *Under the assumptions (2.5)-(2.10), there exists a positive constant τ_{**} such that when $\tau < \tau_{**}$ the finite element system (2.1)-(2.4) admits a unique solution $(P_h^n, \mathbf{U}_h^n, \mathcal{C}_h^n, \mathbf{W}_h^n)$, $n = 1, \dots, N$, which satisfies that*

$$\max_{1 \leq n \leq N} (\|P_h^n - p^n\|_{L^2} + \|\mathbf{U}_h^n - \mathbf{u}^n\|_{L^2} + \|\mathcal{C}_h^n - c^n\|_{L^2}) + \left(\sum_{n=1}^N \tau \|\mathbf{W}_h^n - \mathbf{w}^n\|_{L^2}^2 \right)^{\frac{1}{2}} \leq C_{**}(\tau + h^{r+1}),$$

where C_{**} is some positive constant independent of τ and h .

The proof of Theorem 2.1 is presented in Section 3-5. In Section 3, we introduce an intermediate problem, a system of elliptic interface problems, whose finite element solution coincides with $(P_h^n, \mathbf{U}_h^n, \mathcal{C}_h^n, \mathbf{W}_h^n)$, $n = 0, 1, 2, \dots$. Then we prove that the solution of the system of elliptic interface problems is piecewise smooth enough in each subdomain separated by the interfaces, and the piecewise regularity is uniform with respect to τ (as $\tau \rightarrow 0$). In Section 4, we present error estimates for the finite element discretization of the elliptic interface problems and prove the boundedness of the finite element solution based on the error estimates. In Section 5, we prove the error estimates in Theorem 2.1 based on the boundedness of the finite element solution. Our analysis in Section 3 relies on the following two lemmas concerning the piecewise regularity of some elliptic and parabolic interface problems, which are generalizations of the results in [4, 9, 11, 22] to problems with nonsmooth coefficients, with more precise dependence on the regularity of the coefficients. The proofs of the lemmas are given in Section 6.

Lemma 2.1 *If $A_{ij} \in \overline{H}^2$ satisfies that $K^{-1}|\xi|^2 \leq \sum_{i,j=1}^d A_{ij}(x)\xi_i\xi_j \leq K|\xi|^2$ for $x \in \Omega$ and $\xi \in \mathbb{R}^d$, and $\phi \in H^1$ is a solution of*

$$\begin{cases} -\nabla \cdot (A\nabla\phi) = f & \text{in } \Omega, \\ [\phi] = 0, \quad [A\nabla\phi \cdot \mathbf{n}] = g_m & \text{on } \Gamma_m, \quad m = 1, \dots, M, \\ A\nabla\phi \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.11)$$

then

$$\|\phi\|_{\overline{H}^k} \leq C \left(\|f\|_{\overline{H}^{k-2}} + \sum_{m=1}^M \|g_m\|_{H^{k-3/2}(\Gamma_m)} \right), \quad k = 2, 3. \quad (2.12)$$

where the constant C is independent of τ .

Lemma 2.2 *Suppose that*

$$\begin{aligned} \max_{0 \leq n \leq N-1} \sum_{i,j=1}^d (\|A_{ij}^{n+1}\|_{\overline{H}^2} + d_{n,0} \|D_\tau A_{ij}^{n+1}\|_{L^2}) &\leq K, \\ K^{-1}|\xi|^2 &\leq \sum_{i,j=1}^d A_{ij}^{n+1}(x)\xi_i\xi_j \leq K|\xi|^2, \quad \forall x \in \Omega, \quad \xi \in \mathbb{R}^d, \end{aligned}$$

and $\phi^{n+1} \in H^1$, $n = 0, 1, \dots, N-1$, are solutions of

$$\begin{cases} \Phi D_\tau \phi^{n+1} - \nabla \cdot (A^{n+1} \nabla \phi^{n+1}) = f^{n+1} - \nabla \cdot \mathbf{g}^{n+1} & \text{in } \Omega, \\ [\phi^{n+1}] = 0, \quad [A^{n+1} \nabla \phi^{n+1} \cdot \mathbf{n}] = [\mathbf{g}^{n+1} \cdot \mathbf{n}] & \text{on } \Gamma_m, \quad m = 1, \dots, M, \\ A^{n+1} \nabla \phi^{n+1} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \\ \phi^0 = 0 & \text{in } \Omega. \end{cases} \quad (2.13)$$

Then we have

$$\max_{0 \leq n \leq m} \|\phi^{n+1}\|_{\overline{H}^1}^2 + \sum_{n=0}^m \tau \|\phi^{n+1}\|_{\overline{H}^2}^2 \quad (2.14)$$

$$\leq C\|g^{m+1}\|_{L^2}^2 + C_\epsilon \sum_{n=0}^m \tau (\|f^{n+1}\|_{L^2}^2 + \|\mathbf{g}^{n+1}\|_{\overline{H}^1}^2 + \|\phi^{n+1}\|_{\overline{H}^1}^2) + \epsilon \sum_{n=0}^m \tau \|D_\tau \mathbf{g}^{n+1}\|_{L^2}^2 d_{n,0},$$

where the constant C_ϵ (dependent on ϵ) is independent of τ , and $d_{n,0} = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } 1 \leq n \leq N-1. \end{cases}$

In the rest part of this paper, we denote by C a generic positive constant and by ϵ a small generic positive constant, which are independent of n , τ and h .

3 The linearized PDEs: a system of elliptic interface problems

We introduce (P^n, \mathcal{C}^n) , $n = 0, 1, 2, \dots$, as the solution of an iterative system of linear elliptic interface problems:

$$\begin{cases} -\nabla \cdot \left(\frac{k(x)}{\mu(\mathcal{C}^n)} \nabla P^n \right) = q_I^n - q_P^n & \text{in } \Omega_m, \\ [P^n] = 0, \quad \left[\frac{k(x)}{\mu(\mathcal{C}^n)} \nabla P^n \cdot \mathbf{n} \right] = 0 & \text{on } \Gamma_m, \\ \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla P^n \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

$$\begin{cases} \Phi(x) D_\tau \mathcal{C}^{n+1} - \nabla \cdot (D(\mathbf{U}^n, x) \nabla \mathcal{C}^{n+1}) + \mathbf{U}^n \cdot \nabla \mathcal{C}^{n+1} = \hat{c}^{n+1} q_I^{n+1} - \mathcal{C}^{n+1} q_I^{n+1} & \text{in } \Omega_m, \\ [\mathcal{C}^{n+1}] = 0, \quad [D(\mathbf{U}^n, x) \nabla \mathcal{C}^{n+1} \cdot \mathbf{n}] = 0 & \text{on } \Gamma_m, \\ D(\mathbf{U}^n, x) \nabla \mathcal{C}^{n+1} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.2)$$

with the initial condition $\mathcal{C}^0 = c_0$ and the normalization condition $\int_\Omega P^n dx = 0$. Existence and uniqueness of the solution for the linear elliptic interface problems (3.1)-(3.2) follow iteratively, and it is easy to see that $P^0 = p^0$ and $\mathbf{U}^0 = \mathbf{u}^0$ at the initial time step. With this definition, the fully discrete solution $(P_h^n, \mathbf{U}_h^n, \mathcal{C}_h^n, \mathbf{W}_h^n)$, $n = 0, 1, 2, \dots$, defined in (2.1)-(2.4) can also be viewed as the finite element solution of (3.1)-(3.2).

In this section, we establish the uniform piecewise regularity of (P^n, \mathcal{C}^n) with respect to τ . The following proposition is the main result of this section.

Proposition 3.1 *There exists a positive constant τ_0 such that when $\tau < \tau_0$, we have*

$$\begin{aligned} & \|P^n\|_{\overline{H}^3} + \|\mathbf{U}^n\|_{\overline{H}^2} + \|\mathcal{C}^n\|_{\overline{H}^3} + \|D_\tau \mathcal{C}^n\|_{\overline{H}^1} \\ & + \left(\sum_{n=1}^N \tau \|D_\tau \mathbf{U}^n\|_{\overline{H}^2}^2 \right)^{\frac{1}{2}} + \left(\sum_{n=1}^N \tau \|D_\tau \mathcal{C}^n\|_{\overline{H}^2}^2 \right)^{\frac{1}{2}} \leq C. \end{aligned} \quad (3.3)$$

The importance of this proposition is that the constant C does not depend on τ , which is the key to our error estimates in the next section.

Proof of Proposition 3.1 Let $e_p^n = P^n - p^n$, $e_c^n = \mathcal{C}^n - c^n$ and $e_u^n = \mathbf{U}^n - \mathbf{u}^n$. Comparing (3.1)-(3.2) with (1.1)-(1.2), we see that e_p^n , e_c^n and e_u^n satisfy the equations

$$\begin{cases} -\nabla \cdot \left(\frac{k(x)}{\mu(\mathcal{C}^n)} \nabla e_p^n \right) = \nabla \cdot \left[\left(\frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^n \right], & \text{in } \Omega_m, \\ [e_p^n] = 0, \quad -\left[\frac{k(x)}{\mu(\mathcal{C}^n)} \nabla e_p^n \cdot \mathbf{n} \right] = \left[\left(\frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^n \cdot \mathbf{n} \right] & \text{on } \Gamma_m, \\ \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla e_p^n \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.4)$$

$$\begin{cases} \Phi(x) D_\tau e_c^{n+1} - \nabla \cdot (D(\mathbf{U}^n, x) \nabla e_c^{n+1}) \\ = -\mathbf{U}^n \cdot \nabla e_c^{n+1} + \nabla \cdot ((D(\mathbf{U}^n, x) - D(\mathbf{u}^{n+1}, x)) \nabla c^{n+1}) \\ - (\mathbf{U}^n - \mathbf{u}^n) \cdot \nabla c^{n+1} - e_c^{n+1} q_I^{n+1} + \mathcal{E}^{n+1}, & \text{in } \Omega_m, \\ [e_c^{n+1}] = 0, \quad [D(\mathbf{U}^n, x) \nabla e_c^{n+1} \cdot \mathbf{n}] = [(D(\mathbf{u}^{n+1}, x) - D(\mathbf{U}^n, x)) \nabla c^{n+1} \cdot \mathbf{n}] & \text{on } \Gamma_m, \\ D(\mathbf{U}^n, x) \nabla e_c^{n+1} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \\ e_c^0 = 0 & \text{in } \Omega, \end{cases} \quad (3.5)$$

and

$$e_u^n = -\frac{k(x)}{\mu(\mathcal{C}^n)} \nabla e_p^n - \left(\frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^n, \quad (3.6)$$

respectively, where $\mathcal{E}^{n+1} = \Phi(\partial_t c^{n+1} - D_\tau c^{n+1}) + (\mathbf{u}^{n+1} - \mathbf{u}^n) \cdot \nabla c^{n+1}$ denotes the truncation error due to the time discretization. From the regularity assumption for c in (2.10) we can see that

$$\|\mathcal{E}^{n+1}\|_{L^2} \leq C\tau^{1/2}, \quad \sum_{n=0}^{N-1} \tau \|\mathcal{E}^{n+1}\|_{L^2}^2 \leq C\tau^2. \quad (3.7)$$

Integrating (3.4) against e_p^n , we get

$$\|\nabla e_p^n\|_{L^2} \leq \left\| \left(\frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^n \right\|_{L^2} \leq C \|e_c^n\|_{L^2} \quad (3.8)$$

which together with (3.6) gives

$$\|e_u^n\|_{L^2} \leq C \|\nabla e_p^n\|_{L^2} + C \|e_c^n\|_{L^2} \leq C \|e_c^n\|_{L^2}. \quad (3.9)$$

Then we integrate (3.5) against e_c^{n+1} and obtain

$$\begin{aligned} & D_\tau \left(\frac{1}{2} \|\sqrt{\Phi} e_c^{n+1}\|_{L^2}^2 \right) + \|\sqrt{D(\mathbf{U}^n, x)} \nabla e_c^{n+1}\|_{L^2}^2 \\ & \leq C \|e_c^{n+1}\|_{L^2}^2 \|q_I^{n+1} - q_P^{n+1}\|_{L^\infty} + C (\|e_u^n\|_{L^2} + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_{L^2}) \|\nabla e_c^{n+1}\|_{L^2} \|\nabla c^{n+1}\|_{L^\infty} \\ & \quad + C \|e_u^n\|_{L^2} \|e_c^{n+1}\|_{L^2} \|\nabla c^{n+1}\|_{L^\infty} + C \|e_c^{n+1}\|_{L^2}^2 \|q_I^{n+1}\|_{L^\infty} + C \|\mathcal{E}^{n+1}\|_{L^2} \|e_c^{n+1}\|_{L^2} \end{aligned}$$

$$\leq \frac{1}{2} \|\nabla e_c^{n+1}\|_{L^2}^2 + C(\|e_c^{n+1}\|_{L^2}^2 + \|e_c^n\|_{L^2}^2 + \|\mathcal{E}^{n+1}\|_{L^2}^2 + \|D_\tau \mathbf{u}\|_{L^2}^2 \tau^2), \quad (3.10)$$

where we have used the inequality

$$|(\mathbf{U}^n \cdot \nabla e_c^{n+1}, e_c^{n+1})| = |(\nabla \cdot \mathbf{U}^n, |e_c^{n+1}|^2)| \leq \|e_c^{n+1}\|_{L^2}^2 \|q_I^{n+1} - q_P^{n+1}\|_{L^\infty}.$$

By applying Gronwall's inequality to (3.10), we see that there exists a positive constant τ_1 such that when $\tau < \tau_1$ there holds

$$\max_{1 \leq n \leq N} \|e_c^n\|_{L^2}^2 + \sum_{n=1}^N \tau \|\nabla e_c^n\|_{L^2}^2 \leq C\tau^2. \quad (3.11)$$

The last inequality, together with (3.8)-(3.9), implies that

$$\max_{1 \leq n \leq N} \|D_\tau e_c^n\|_{L^2}^2 + \sum_{n=1}^N \tau \|D_\tau e_c^n\|_{L^6}^2 \leq C, \quad (3.12)$$

$$\max_{1 \leq n \leq N} (\|e_u^n\|_{L^2} + \|\nabla e_p^n\|_{L^2}) \leq C\tau. \quad (3.13)$$

Let $d_{n,0}$ be the constant defined in Lemma 2.2. We proceed with a mathematical induction on

$$\|\mathcal{C}^n\|_{\overline{H}^2} + \|\mathbf{U}^n\|_{\overline{H}^2} + d_{n,0} \|D_\tau \mathbf{U}^n\|_{L^2} \leq \max_{0 \leq n \leq N} (\|\mathcal{C}^n\|_{\overline{H}^2} + \|\mathbf{u}^n\|_{\overline{H}^2} + d_{n,0} \|D_\tau \mathbf{u}^n\|_{L^2}) + 1, \quad (3.14)$$

which clearly holds when $n = 0$ (as $\mathcal{C}^0 = c^0$ and $\mathbf{U}^0 = \mathbf{u}^0$). We shall assume that the above inequality holds for $0 \leq n \leq k$ and prove that it also holds for $n = k + 1$.

With (3.14), we can apply Lemma 2.1 to (3.4) for $0 \leq n \leq k$ and obtain

$$\begin{aligned} \|e_p^n\|_{\overline{H}^2} &\leq C \left\| \nabla \cdot \left[\left(\frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^n \right] \right\|_{\overline{L}^2} + C \left\| \left[\left(\frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^n \right] \right\|_{H^{1/2}(\Gamma)} \leq C \|e_c^n\|_{\overline{H}^1}, \\ \|e_p^n\|_{\overline{H}^3} &\leq C \left\| \nabla \cdot \left[\left(\frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^n \right] \right\|_{\overline{H}^1} + C \left\| \left[\left(\frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^n \right] \right\|_{H^{3/2}(\Gamma)} \leq C \|e_c^n\|_{\overline{H}^2}, \end{aligned}$$

and from (3.6) we see that

$$\|e_u^n\|_{\overline{H}^1} \leq C \|e_c^n\|_{\overline{H}^1}, \quad (3.15)$$

$$\|e_u^n\|_{\overline{H}^2} \leq C \|e_c^n\|_{\overline{H}^2}. \quad (3.16)$$

As a consequence, by the Sobolev embedding inequality, we have

$$\|\nabla e_p^n\|_{L^\infty} \leq C \|e_p^n\|_{\overline{H}^3} \leq C \quad \text{and} \quad \|e_u^n\|_{\overline{H}^2} \leq C, \quad \text{for } 0 \leq n \leq k.$$

Applying the difference operator D_τ to the equation (3.5), we obtain

$$\begin{cases} -\nabla \cdot \left(\frac{k(x)}{\mu(\mathcal{C}^{n+1})} \nabla D_\tau e_p^{n+1} \right) = \nabla \cdot \mathbf{f}^{n+1}, & \text{in } \Omega, \\ [D_\tau e_p^{n+1}] = 0, \quad -\left[\frac{k(x)}{\mu(\mathcal{C}^{n+1})} \nabla D_\tau e_p^{n+1} \cdot \mathbf{n} \right] = [\mathbf{f}^{n+1} \cdot \mathbf{n}] & \text{on } \Gamma_m, \\ \frac{k(x)}{\mu(\mathcal{C}^{n+1})} \nabla D_\tau e_p^{n+1} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.17)$$

where

$$\mathbf{f}^{n+1} = D_\tau \left(\frac{k(x)}{\mu(\mathcal{C}^{n+1})} \right) \nabla e_p^n + D_\tau \left[\left(\frac{k(x)}{\mu(\mathcal{C}^{n+1})} - \frac{k(x)}{\mu(c^{n+1})} \right) \nabla p^{n+1} \right].$$

For $0 \leq n \leq k$, from (3.2) and (3.5) we can derive that

$$\begin{aligned} \|D_\tau \mathcal{C}^{n+1}\|_{L^2} &\leq C \|\mathcal{C}^{n+1}\|_{\overline{H}^2} + C \|\mathbf{U}^n\|_{\overline{H}^2} \\ &\leq C \|e_c^{n+1}\|_{\overline{H}^2} + \|c^{n+1}\|_{\overline{H}^2} + C \|\mathbf{U}^n\|_{\overline{H}^2} \\ &\leq C + C \|e_c^{n+1}\|_{\overline{H}^2}, \\ \|D_\tau e_c^{n+1}\|_{L^2} &\leq C \|e_c^{n+1}\|_{\overline{H}^2} + C \|e_u^n\|_{\overline{H}^2} + C \|\mathcal{E}^{n+1}\|_{L^2} \\ &\leq C \|e_c^{n+1}\|_{\overline{H}^2} + C \|e_c^n\|_{\overline{H}^2} + C \|\mathcal{E}^{n+1}\|_{L^2}, \end{aligned}$$

from (3.17) we see that

$$\begin{aligned} \|\nabla D_\tau e_p^{n+1}\|_{L^2} &\leq C \|\mathbf{f}^{n+1}\|_{L^2} \\ &\leq C \|D_\tau \mathcal{C}^{n+1}\|_{L^2} \|\nabla e_p^n\|_{L^\infty} + C \|D_\tau e_c^{n+1}\|_{L^2} + C \|e_c^{n+1}\|_{L^2} \|D_\tau c^n\|_{L^\infty} \\ &\leq C \|\nabla e_p^n\|_{L^\infty} + C \|e_c^{n+1}\|_{\overline{H}^2} \|\nabla e_p^n\|_{L^\infty} + C \|e_c^{n+1}\|_{\overline{H}^2} + C \|e_c^n\|_{\overline{H}^2} + C \|\mathcal{E}^{n+1}\|_{L^2} \\ &\leq C \|e_c^n\|_{\overline{H}^2} + C \|e_c^{n+1}\|_{\overline{H}^2} + C \|\mathcal{E}^{n+1}\|_{L^2}, \end{aligned}$$

and from (3.6) we derive that

$$\begin{aligned} \|D_\tau e_u^{n+1}\|_{L^2} &\leq C (\|k(x)/\mu(\mathcal{C}^{n+1})\|_{L^\infty} \|\nabla D_\tau e_p^{n+1}\|_{L^2} + \|D_\tau \mathcal{C}^{n+1}\|_{L^2} \|\nabla e_p^n\|_{L^\infty}) \\ &\quad + C (\|e_c^n\|_{L^\infty} \|D_\tau c^{n+1}\|_{L^2} + \|k(x)/\mu(\mathcal{C}^{n+1})\|_{L^\infty} \|D_\tau e_c^{n+1}\|_{L^2}) \|\nabla p^{n+1}\|_{L^\infty} \\ &\quad + C \|e_c^{n+1}\|_{L^\infty} \|\nabla D_\tau p^{n+1}\|_{L^2} \\ &\leq C (\|\nabla D_\tau e_p^{n+1}\|_{L^2} + \|e_p^n\|_{\overline{H}^3} + \|e_c^{n+1}\|_{\overline{H}^2} + \|e_c^n\|_{\overline{H}^2} + \|D_\tau e_c^{n+1}\|_{L^2}) \\ &\leq C (\|e_c^{n+1}\|_{\overline{H}^2} + \|e_c^n\|_{\overline{H}^2} + \|\mathcal{E}^{n+1}\|_{L^2}). \end{aligned} \tag{3.18}$$

With (3.14)-(3.18), we let

$$f^{n+1} = -\mathbf{U}^n \cdot \nabla e_c^{n+1} + \nabla \cdot ((D(\mathbf{U}^n, x) - D(\mathbf{u}^{n+1}, x)) \nabla c^{n+1}) - (\mathbf{U}^n - \mathbf{u}^n) \cdot \nabla c^{n+1} - e_c^{n+1} q_I^{n+1} + \mathcal{E}^{n+1},$$

and apply Lemma 2.2 to (3.5) for $0 \leq n \leq k$. Then we derive that, for $0 \leq m \leq k$,

$$\begin{aligned} &\max_{0 \leq n \leq m} \|e_c^{n+1}\|_{\overline{H}^1}^2 + \sum_{n=0}^m \tau \|e_c^{n+1}\|_{\overline{H}^2}^2 \\ &\leq C \max_{0 \leq n \leq m} \|(D(\mathbf{u}^n, x) - D(\mathbf{u}^{n+1}, x)) \nabla c^{n+1}\|_{L^2}^2 \\ &\quad + \epsilon \sum_{n=0}^m \tau \|D_\tau ((D(\mathbf{U}^n, x) - D(\mathbf{u}^{n+1}, x)) \nabla c^{n+1})\|_{L^2}^2 d_{n,0} \\ &\quad + C_\epsilon \sum_{n=0}^m \tau (\|f^{n+1}\|_{L^2}^2 + \|(D(\mathbf{U}^n, x) - D(\mathbf{u}^{n+1}, x)) \nabla c^{n+1}\|_{\overline{H}^1} + \|e_c^{n+1}\|_{\overline{H}^1}^2) \\ &\leq C \tau^2 + \epsilon \sum_{n=0}^m \tau (\|e_u^n\|_{L^\infty}^2 + \|D_\tau e_u^n\|_{L^2}^2 + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_{L^\infty}^2 + \|D_\tau (D(\mathbf{u}^{n+1}, x) - D(\mathbf{u}^n, x))\|_{L^2}^2) d_{n,0} \end{aligned}$$

$$\begin{aligned}
& + C_\epsilon \sum_{n=0}^m \left(\|\mathbf{U}^n\|_{L^\infty}^2 \|\nabla e_c^{n+1}\|_{L^2}^2 + (\|e_u^n\|_{\overline{H}^1}^2 + \tau^2 \|D_\tau \mathbf{u}^{n+1}\|_{\overline{H}^1}^2) \|\nabla c^{n+1}\|_{L^\infty}^2 \right. \\
& \left. + (\|e_u^n\|_{L^6}^2 + \tau^2 \|D_\tau \mathbf{u}^{n+1}\|_{L^6}^2) \|c^{n+1}\|_{\overline{W}^{2,3}}^2 + \|e_u^n\|_{L^2}^2 \|\nabla c^{n+1}\|_{L^\infty}^2 + \|e_c^{n+1}\|_{\overline{H}^1}^2 + \|\mathcal{E}^{n+1}\|_{L^2}^2 \right) \\
& \leq C\tau^2 + \epsilon \sum_{n=0}^m \tau (\|e_c^n\|_{\overline{H}^2}^2 + \|e_c^{n-1}\|_{\overline{H}^2}^2 + \|\mathcal{E}^n\|_{L^2}^2) d_{n,0}
\end{aligned} \tag{3.19}$$

$$+ C_\epsilon \sum_{n=0}^m \tau (\|e_c^{n+1}\|_{\overline{H}^1}^2 + \|e_c^n\|_{\overline{H}^1}^2 + \|\mathcal{E}^{n+1}\|_{L^2}^2 + \tau^2 \|D_\tau \mathbf{u}^{n+1}\|_{\overline{H}^1}^2), \tag{3.20}$$

which reduces to

$$\max_{0 \leq n \leq m} \|e_c^{n+1}\|_{\overline{H}^1}^2 + \sum_{n=0}^m \tau \|e_c^{n+1}\|_{\overline{H}^2}^2 \leq C\tau^2 + \sum_{n=0}^m \tau (\|e_c^{n+1}\|_{\overline{H}^1}^2 + \|e_c^n\|_{\overline{H}^1}^2).$$

By applying Gronwall's inequality, there exists a positive constant τ_3 such that when $\tau < \tau_3$

$$\max_{1 \leq n \leq k} \|e_c^{n+1}\|_{\overline{H}^1}^2 + \sum_{n=1}^k \tau \|e_c^{n+1}\|_{\overline{H}^2}^2 \leq C\tau^2. \tag{3.21}$$

From the last inequality we see that

$$\begin{aligned}
& \max_{0 \leq n \leq k} (\|D_\tau c^{n+1}\|_{\overline{H}^1}^2 + \|c^{n+1}\|_{\overline{H}^2}^2) + \sum_{n=0}^k \tau \|D_\tau c^{n+1}\|_{\overline{H}^2}^2 \leq C, \\
& \max_{0 \leq n \leq k} \|e_c^{n+1}\|_{\overline{H}^2} \leq C\tau^{1/2},
\end{aligned} \tag{3.22}$$

and from (3.18) we see that

$$\|D_\tau e_u^{k+1}\|_{L^2} \leq C(\|e_c^{k+1}\|_{\overline{H}^2} + \|e_c^k\|_{\overline{H}^2} + \|\mathcal{E}^{k+1}\|_{L^2}) \leq C\tau^{1/2}.$$

With $\max_{0 \leq n \leq k} \|c^{n+1}\|_{\overline{H}^2} \leq C$ given by (3.22), we can apply Lemma 2.1 to (3.4) again and obtain

$$\begin{aligned}
& \|e_p^{k+1}\|_{\overline{H}^3} + \|e_u^{k+1}\|_{\overline{H}^2} \leq C\|e_c^{k+1}\|_{\overline{H}^2} \leq C\tau^{1/2}, \\
& \sum_{n=0}^k \tau \|e_u^{n+1}\|_{\overline{H}^2}^2 \leq C \sum_{n=0}^k \tau \|e_c^{n+1}\|_{\overline{H}^2}^2 \leq C\tau^2,
\end{aligned}$$

and so

$$\max_{0 \leq n \leq k} (\|P^{n+1}\|_{\overline{H}^3}^2 + \|\mathbf{U}^{n+1}\|_{\overline{H}^2}^2) + \sum_{n=0}^k \tau \|D_\tau \mathbf{U}^{n+1}\|_{\overline{H}^2}^2 \leq C, \tag{3.23}$$

The last five inequalities imply that there exists a positive constant τ_4 such that when $\tau < \tau_4$ we have

$$\|e_c^{k+1}\|_{\overline{H}^2} + \|e_u^{k+1}\|_{\overline{H}^2} + \|D_\tau e_u^{k+1}\|_{L^2} \leq 1.$$

The mathematical induction on (3.14) is completed. Thus (3.22)-(3.23) hold for $k = N - 1$ with the same constant C , provided $\tau < \tau_0 := \min(\tau_1, \tau_2, \tau_3, \tau_4)$.

With the regularity $\max_{0 \leq n \leq N} \|D_\tau \mathcal{C}^n\|_{\overline{H}^1} \leq C$, as shown in (3.22), by applying Lemma 2.1 to (3.2) we obtain

$$\max_{1 \leq n \leq N} \|\mathcal{C}^n\|_{\overline{H}^3} \leq C. \quad (3.24)$$

The proof of Proposition 3.1 is completed. ■

4 Boundedness of the \mathbf{U}_h^n

Based on the finite element discretization of the elliptic interface problems, we prove the following proposition in this section.

Proposition 4.1 *There exist positive constants τ_* and h_* such that when $\tau < \tau_*$ and $h < h_*$, the finite element system (2.1)-(2.4) admits a unique solution $(P_h^n, \mathbf{U}_h^n, \mathbf{W}_h^n, \mathcal{C}_h^n)$, $n = 1, \dots, N$, such that*

$$\max_{1 \leq n \leq N} \|\mathbf{U}_h^n\|_{L^\infty} \leq C.$$

Before we prove this proposition, we define some notations below. Let L_h denote the piecewise linear Lagrange interpolation operator onto the finite element space S_h^r . Let Π_h denote the L^2 projection onto the finite element space S_h^r , i.e.

$$(\phi - \Pi_h \phi, \chi_h) = 0, \quad \forall \phi \in L^2, \chi_h \in S_h^r$$

and let Π_h^Γ denote the L^2 projection onto the finite element space $S_h^r(\Gamma)$ satisfying

$$(\phi - \Pi_h^\Gamma \phi, \chi_h)_\Gamma = 0, \quad \forall \phi \in L^2(\Gamma), \chi_h \in S_h^r(\Gamma).$$

Let $Q_h : \mathbf{H}_\Gamma^1 \rightarrow \mathbf{H}_h^r$ be a projection satisfying (see [16, 33] for the construction of such a projection operator)

$$(\nabla \cdot (\mathbf{v} - Q_h \mathbf{v}), \chi_h) = 0, \quad \forall \chi_h \in S_h^r, \mathbf{v} \in \mathbf{H}_\Gamma^1, \quad (4.1)$$

$$\int_{\tilde{e}_j} (\mathbf{v} - Q_h \mathbf{v}) \cdot \mathbf{n} \chi_h \, ds = 0, \quad \forall \chi_h \in S_h^r, \mathbf{v} \in \mathbf{H}_\Gamma^1, \quad (4.2)$$

for any edge \tilde{e}_j in the triangulation. Then we have

$$\|\mathbf{v} - Q_h \mathbf{v}\|_{L^2} + \|\mathbf{v} - Q_h \mathbf{v}\|_{L^2(\Gamma_m)} h^{1/2} + \|\nabla \cdot (\mathbf{v} - Q_h \mathbf{v})\|_{L^2} h \leq C \|\mathbf{v}\|_{\overline{H}^k} h^k, \quad k = 1, 2, \dots$$

Let $\mathbf{W}^{n+1} = -D(\mathbf{U}^n, x) \nabla \mathcal{C}^{n+1} \in \mathbf{H}_\Gamma^1$ and, for any fixed integer $n \geq -1$, let $(\overline{\mathcal{C}}_h^{n+1}, \overline{\mathbf{W}}_h^{n+1}) \in S_h^r \times \mathbf{H}_h^r$ be the finite element solution of the equation

$$\begin{cases} (\nabla \cdot (\overline{\mathbf{W}}_h^{n+1} - \mathbf{W}^{n+1}), \chi_h) = 0, & \forall \chi_h \in S_h^r, \\ (D(\mathbf{U}^n, x)^{-1} (\overline{\mathbf{W}}_h^{n+1} - \mathbf{W}^{n+1}), \mathbf{v}_h) = (\overline{\mathcal{C}}_h^{n+1} - \mathcal{C}^{n+1}, \nabla \cdot \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{H}_h^r, \end{cases} \quad (4.3)$$

with $\int_{\Omega}(\bar{\mathcal{C}}_h^{n+1} - \mathcal{C}^{n+1})dx = 0$ for the uniqueness of solution, where we define $\mathbf{U}^{-1} := \mathbf{U}^0$. The pair $(\bar{\mathcal{C}}_h^{n+1}, \bar{\mathbf{W}}_h^{n+1})$ can be viewed as the Ritz projection of $(\mathcal{C}^{n+1}, \mathbf{W}^{n+1})$ by the mixed FEM.

We require $\tau < \tau_0$ so that Proposition 3.1 holds. With the regularity of \mathcal{C}^{n+1} and \mathbf{W}^{n+1} given in Proposition 3.1, by the theory of mixed FEM for linear elliptic equations [16, 33], we have

$$\|\phi - \Pi_h \phi\|_{L^2} \leq Ch^2 \|\phi\|_{\bar{H}^2}, \quad \forall \phi \in \bar{H}^2, \quad (4.4)$$

$$\|\phi - \Pi_h^\Gamma \phi\|_{L^2(\Gamma)} \leq Ch^{k+1/2} \|\phi\|_{H^{k+1/2}(\Gamma)}, \quad \forall \phi \in \bar{H}^{k+1/2}(\Gamma) \quad \text{with } 1 \leq k \leq r, \quad (4.5)$$

$$\|\mathbf{W}^{n+1} - L_h \mathbf{W}^{n+1}\|_{L^2} + \|\mathbf{W}^{n+1} - \bar{\mathbf{W}}_h^{n+1}\|_{L^2} \leq Ch^2, \quad (4.6)$$

$$\|\mathcal{C}^{n+1} - \bar{\mathcal{C}}_h^{n+1}\|_{L^2} \leq Ch^2. \quad (4.7)$$

Therefore, by the inverse inequality, we have

$$\begin{aligned} & \|Q_h \mathbf{W}^{n+1} - L_h \mathbf{W}^{n+1}\|_{L^\infty} + \|\bar{\mathbf{W}}_h^{n+1} - L_h \mathbf{W}^{n+1}\|_{L^\infty} \\ & \leq Ch^{-d/2} \|Q_h \mathbf{W}^{n+1} - L_h \mathbf{W}^{n+1}\|_{L^2} \leq Ch^{2-d/2}, \end{aligned}$$

which implies the existence of a positive constant h_1 such that when $h < h_1$ there holds

$$\|Q_h \mathbf{W}^{n+1}\|_{L^\infty} + \|\bar{\mathbf{W}}_h^{n+1}\|_{L^\infty} \leq 2\|L_h \mathbf{W}^{n+1}\|_{L^\infty} + 1 \leq C. \quad (4.8)$$

Moreover, we need the following two lemmas in the proof of Proposition 4.1.

Lemma 4.1 *Under the regularity of \mathcal{C}^{n+1} and \mathbf{U}^{n+1} proved in Proposition 3.1, we have*

$$\left(\sum_{n=0}^{N-1} \tau \|D_\tau(\mathcal{C}^{n+1} - \bar{\mathcal{C}}_h^{n+1})\|_{L^2}^2 \right)^{1/2} \leq Ch^2. \quad (4.9)$$

Proof From (4.3) we derive that

$$(\nabla \cdot (D_\tau \bar{\mathbf{W}}_h^{n+1} - D_\tau \mathbf{W}^{n+1}), \chi_h) = 0, \quad \forall \chi_h \in S_h^r, \quad (4.10)$$

$$\begin{aligned} & (D(\mathbf{U}^n, x)^{-1} (D_\tau \bar{\mathbf{W}}_h^{n+1} - D_\tau \mathbf{W}^{n+1}), \mathbf{v}_h) + (D_\tau D(\mathbf{U}^n, x)^{-1} (\bar{\mathbf{W}}_h^n - \mathbf{W}^n), \mathbf{v}_h) \\ & = (D_\tau (\bar{\mathcal{C}}_h^{n+1} - \Pi_h \mathcal{C}^{n+1}), \nabla \cdot \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{H}_h^r, \end{aligned} \quad (4.11)$$

where (4.10) implies that $\nabla \cdot (D_\tau \bar{\mathbf{W}}_h^{n+1} - Q_h D_\tau \mathbf{W}^{n+1}) = 0$. By choosing $\mathbf{v}_h = D_\tau \bar{\mathbf{W}}_h^{n+1} - Q_h D_\tau \mathbf{W}^{n+1}$ in (4.11), we derive that

$$\|D_\tau \bar{\mathbf{W}}_h^{n+1} - Q_h D_\tau \mathbf{W}^{n+1}\|_{L^2} \leq C \|D_\tau \mathbf{W}^{n+1} - Q_h D_\tau \mathbf{W}^{n+1}\|_{L^2} + C \|\mathbf{W}^{n+1} - \bar{\mathbf{W}}_h^{n+1}\|_{L^2},$$

and so, by the inverse inequality,

$$\|\nabla \cdot (D_\tau \bar{\mathbf{W}}_h^{n+1} - Q_h D_\tau \mathbf{W}^{n+1})\|_{L^2} \leq Ch^{-1} \|D_\tau \bar{\mathbf{W}}_h^{n+1} - Q_h D_\tau \mathbf{W}^{n+1}\|_{L^2} \leq C \|D_\tau \mathbf{W}^{n+1}\|_{H^1} + C.$$

Let $\mathbf{v} = -D(\mathbf{U}^n, x) \nabla g$, where g is the solution of the elliptic interface problem

$$\begin{cases} -\nabla \cdot (D(\mathbf{U}^n, x) \nabla g) = D_\tau (\bar{\mathcal{C}}_h^{n+1} - \Pi_h \mathcal{C}^{n+1}) & \text{in } \Omega_m, \\ [g] = 0, \quad [D(\mathbf{U}^n, x) \nabla g \cdot \mathbf{n}] = 0 & \text{on } \Gamma_m, \\ D(\mathbf{U}^n, x) \nabla g \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Substituting $\mathbf{v}_h = Q_h \mathbf{v}$ into (4.11), we obtain

$$\begin{aligned}
& \|D_\tau(\mathcal{C}^{n+1} - \bar{\mathcal{C}}_h^{n+1})\|_{L^2}^2 \\
&= (D(\mathbf{U}^n, x)^{-1} D_\tau(\bar{\mathbf{W}}_h^{n+1} - \mathbf{W}^{n+1}), \mathbf{v}) \\
&\quad + (D(\mathbf{U}^n, x)^{-1} D_\tau(\bar{\mathbf{W}}_h^{n+1} - \mathbf{W}^{n+1}), \mathbf{v}_h - \mathbf{v}) + (D_\tau D(\mathbf{U}^n, x)^{-1}(\bar{\mathbf{W}}_h^n - \mathbf{W}^n), \mathbf{v}_h) \\
&= (\nabla \cdot (D_\tau \bar{\mathbf{W}}_h^{n+1} - D_\tau \mathbf{W}^{n+1}), g) - ([(D_\tau \bar{\mathbf{W}}_h^{n+1} - D_\tau \mathbf{W}^{n+1}) \cdot \mathbf{n}], g)_\Gamma \\
&\quad + (D(\mathbf{U}^n, x)^{-1} D_\tau(\bar{\mathbf{W}}_h^{n+1} - \mathbf{W}^{n+1}), \mathbf{v}_h - \mathbf{v}) + (D_\tau D(\mathbf{U}^n, x)^{-1}(\bar{\mathbf{W}}_h^n - \mathbf{W}^n), \mathbf{v}_h) \\
&= (\nabla \cdot (D_\tau \bar{\mathbf{W}}_h^{n+1} - D_\tau \mathbf{W}^{n+1}), g - L_h g) - ([(D_\tau \bar{\mathbf{W}}_h^{n+1} - D_\tau \mathbf{W}^{n+1}) \cdot \mathbf{n}], g - \Pi_h^\Gamma g)_\Gamma \\
&\quad + (D(\mathbf{U}^n, x)^{-1} D_\tau(\bar{\mathbf{W}}_h^{n+1} - \mathbf{W}^{n+1}), \mathbf{v}_h - \mathbf{v}) + (D_\tau D(\mathbf{U}^n, x)^{-1}(\bar{\mathbf{W}}_h^n - \mathbf{W}^n), \mathbf{v}_h) \\
&\leq C \|D_\tau \mathbf{W}^{n+1}\|_{\bar{H}^1} \|g - L_h g\|_{L^2} + C \|[(D_\tau \bar{\mathbf{W}}_h^{n+1} - D_\tau \mathbf{W}^{n+1}) \cdot \mathbf{n}]\|_{L^2(\Gamma)} \|g - \Pi_h^\Gamma g\|_{L^2(\Gamma)} \\
&\quad + \|D_\tau \bar{\mathbf{W}}_h^{n+1} - D_\tau \mathbf{W}^{n+1}\|_{L^2} \|\mathbf{v}_h - \mathbf{v}\|_{L^2} + \|D_\tau D(\mathbf{U}^n, x)^{-1}\|_{L^\infty} \|\bar{\mathbf{W}}_h^n - \mathbf{W}^n\|_{L^2} \|\mathbf{v}_h\|_{L^2} \\
&\leq C \|D_\tau \mathbf{W}^{n+1}\|_{\bar{H}^1} \|g\|_{\bar{H}^2} h^2 + C \|[(D_\tau \bar{\mathbf{W}}_h^{n+1} - Q_h D_\tau \mathbf{W}^{n+1}) \cdot \mathbf{n}]\|_{L^2(\Gamma)} \|g\|_{H^{3/2}(\Gamma)} h^{3/2} \\
&\quad + C \|[(Q_h D_\tau \bar{\mathbf{W}}_h^{n+1} - D_\tau \mathbf{W}^{n+1}) \cdot \mathbf{n}]\|_{L^2(\Gamma)} \|g\|_{H^{3/2}(\Gamma)} h^{3/2} \\
&\quad + C \|D_\tau \mathbf{W}^{n+1}\|_{\bar{H}^1} \|g\|_{\bar{H}^2} h^2 + C \|D_\tau \mathbf{U}^n\|_{L^\infty} \|\mathbf{v}_h\|_{L^2} \\
&\leq C \|D_\tau \mathbf{W}^{n+1}\|_{\bar{H}^1} \|g\|_{\bar{H}^2} h^2 + C h^{-1/2} \|D_\tau \bar{\mathbf{W}}_h^{n+1} - Q_h D_\tau \mathbf{W}^{n+1}\|_{L^2} \|g\|_{H^{3/2}(\Gamma)} h^{3/2} \\
&\quad + C \|D_\tau \mathbf{W}^{n+1}\|_{\bar{H}^1} \|g\|_{H^{3/2}(\Gamma)} h^2 + C \|D_\tau \mathbf{U}^n\|_{\bar{H}^2} \|g\|_{\bar{H}^2} h^2 \\
&\leq C \|D_\tau \mathbf{W}^{n+1}\|_{\bar{H}^1} \|g\|_{\bar{H}^2} h^2 + C \|D_\tau \mathbf{U}^n\|_{\bar{H}^2} \|g\|_{\bar{H}^2} h^2 \\
&\leq C (\|D_\tau \mathcal{C}^{n+1}\|_{\bar{H}^2} + \|D_\tau \mathbf{U}^n\|_{\bar{H}^2}) \|D_\tau(\mathcal{C}^{n+1} - \bar{\mathcal{C}}_h^{n+1})\|_{L^2} h^2,
\end{aligned}$$

which reduces to

$$\|D_\tau(\mathcal{C}^{n+1} - \bar{\mathcal{C}}_h^{n+1})\|_{L^2} \leq C (\|D_\tau \mathcal{C}^{n+1}\|_{\bar{H}^2} + \|D_\tau \mathbf{U}^n\|_{\bar{H}^2}) h^2.$$

The last inequality, together with Proposition 3.1, gives (4.1). \blacksquare

Lemma 4.2 *If $g \in \bar{H}^{k+1}$ with $1 \leq k \leq r$, then*

$$|(g - \Pi_h^\Gamma g, \mathbf{v}_h \cdot \mathbf{n})_{\partial\Omega}| + \sum_{m=1}^M |(g - \Pi_h^\Gamma g, [\mathbf{v}_h \cdot \mathbf{n}])_{\Gamma_m}| \leq C \|g\|_{\bar{H}^{k+1}} \|\mathbf{v}_h\|_{L^2} h^{k+1}.$$

Proof For simplicity, we only prove the 2D case. The 3D case can be proved in the same way. For a triangle T_j on the boundary (or an interface), we denote by e_j its edge with at two vertices on the boundary (or an interface) and denote by \tilde{e}_j the curved edge on the boundary (or an interface). Via a rigid rotation, we assume that e_j is on the x_1 -axis and \tilde{e}_j is parametrized by $(x_1, y(x_1))$. Let $\chi_h \in S_h^r$ be a finite element function whose restriction to e_j coincides with $\mathbf{v}_h \cdot \mathbf{n}$, i.e. $\mathbf{v}_h(x_1, 0) \cdot \mathbf{n}(x_1, 0) = \chi_h(x_1, 0)$, satisfying $\|\chi_h\|_{L^2(T_j)} \leq C \|\mathbf{v}_h\|_{L^2(T_j)}$. Then we have, with $ds = \sqrt{1 + |y'(x_1)|^2} dx_1$,

$$\int_{\tilde{e}_j} (g - \Pi_h^\Gamma g) \mathbf{v}_h \cdot \mathbf{n} ds$$

$$\begin{aligned}
&= \int_{e_j} (g(x_1, y(x_1)) - (\Pi_h^\Gamma g)(x_1, y(x_1))) \mathbf{v}_h(x_1, y(x_1)) \cdot \mathbf{n}(x_1, y(x_1)) \, ds \\
&= \int_{e_j} (g(x_1, y(x_1)) - (\Pi_h^\Gamma g)(x_1, y(x_1))) (\mathbf{v}_h(x_1, y(x_1)) \cdot \mathbf{n}(x_1, y(x_1)) - \mathbf{v}_h(x_1, 0) \cdot \mathbf{n}(x_1, 0)) \, ds \\
&\quad + \int_{e_j} (g(x_1, y(x_1)) - (\Pi_h^\Gamma g)(x_1, y(x_1))) (\chi_h(x_1, 0) - \chi_h(x_1, y(x_1))) \, ds \\
&\leq C \|g - \Pi_h^\Gamma g\|_{L^2(e_j)} (\| \sup_{x_2 \in T_j} \partial_y \mathbf{v}_h(\cdot, x_2) \|_{L^2(e_j)} h^2 + \| \mathbf{v}_h(\cdot, 0) \|_{L^2(e_j)} h + \| \sup_{x_2 \in T_j} \partial_y \chi_h(\cdot, x_2) \|_{L^2(e_j)} h^2) \\
&\leq C \|g\|_{H^{k+1/2}(e_j)} (\| \mathbf{v}_h \|_{L^2(T_j)} + \| \chi_h(\cdot, x_2) \|_{L^2(T_j)}) h^{k+1} \\
&\leq C \|g\|_{H^{k+1}(T_j)} \| \mathbf{v}_h \|_{L^2(T_j)} h^{k+1},
\end{aligned}$$

which implies that

$$\int_{\partial\Omega} (g - \Pi_h^\Gamma g) \mathbf{v}_h \cdot \mathbf{n} \, ds \leq C \sum_j \|g\|_{H^{k+1}(T_j)} \| \mathbf{v}_h \|_{L^2(T_j)} h^{k+1} \leq C \|g\|_{\overline{H}^{k+1}} \| \mathbf{v}_h \|_{L^2} h^{k+1}.$$

The estimate of $\int_{\Gamma_m} (g - \Pi_h^\Gamma g) [\mathbf{v}_h \cdot \mathbf{n}] \, ds$ on an interface Γ_m is similar. ■

Proof of Proposition 4.1 Let $\tau < \tau_0$ so that (4.4)-(4.8) hold. The mixed weak formulation of (3.1)-(3.2) is

$$\left(\frac{\mu(\mathcal{C}^n)}{k(x)} \mathbf{U}^n, \mathbf{v}_h \right) = \left(P^n, \nabla \cdot \mathbf{v}_h \right) - \left(P^n, \mathbf{v}_h \cdot \mathbf{n} \right)_{\partial\Omega} - \sum_{m=1}^M \left(P^n, [\mathbf{v}_h \cdot \mathbf{n}] \right)_{\Gamma_m}, \quad (4.12)$$

$$\left(\nabla \cdot \mathbf{U}^n, \varphi_h \right) = \left(q_I^n - q_P^n, \varphi_h \right), \quad (4.13)$$

$$\left(D(\mathbf{U}^n, x)^{-1} \mathbf{W}^{n+1}, \overline{\mathbf{v}}_h \right) = \left(\mathcal{C}^{n+1}, \nabla \cdot \overline{\mathbf{v}}_h \right) - \left(\mathcal{C}^{n+1}, \overline{\mathbf{v}}_h \cdot \mathbf{n} \right)_{\partial\Omega} - \sum_{m=1}^M \left(\mathcal{C}^{n+1}, [\overline{\mathbf{v}}_h \cdot \mathbf{n}] \right)_{\Gamma_m}, \quad (4.14)$$

$$\begin{aligned}
&\left(\Phi(x) D_\tau \mathcal{C}^{n+1}, \overline{\varphi}_h \right) + \left(\nabla \cdot \mathbf{W}^{n+1}, \overline{\varphi}_h \right) - \left(D(\mathbf{U}^n, x)^{-1} \mathbf{U}^n \cdot \mathbf{W}^{n+1}, \overline{\varphi}_h \right) \\
&= \left(\hat{\mathcal{C}}^{n+1} q_I^{n+1} - \mathcal{C}^{n+1} q_I^{n+1}, \overline{\varphi}_h \right),
\end{aligned} \quad (4.15)$$

for any $\mathbf{v}_h, \overline{\mathbf{v}}_h \in \mathbf{H}_h^r$ and $\varphi_h, \overline{\varphi}_h \in S_h$. The above equations with the finite element system (2.1)-(2.4) imply that

$$\begin{aligned}
&\left(\frac{\mu(\mathcal{C}_h^n)}{k(x)} \mathbf{U}_h^n - \frac{\mu(\mathcal{C}^n)}{k(x)} \mathbf{U}^n, \mathbf{v}_h \right) = \left(P_h^n - \Pi_h P^n, \nabla \cdot \mathbf{v}_h \right) \\
&\quad + \left(P^n - \Pi_h^\Gamma P^n, \mathbf{v}_h \cdot \mathbf{n} \right)_{\partial\Omega} + \sum_{m=1}^M \left(P^n - \Pi_h^\Gamma P^n, [\mathbf{v}_h \cdot \mathbf{n}] \right)_{\Gamma_m},
\end{aligned} \quad (4.16)$$

$$\left(\nabla \cdot (\mathbf{U}_h^n - Q_h \mathbf{U}^n), \varphi_h \right) = 0, \quad (4.17)$$

$$\left(D(\mathbf{U}_h^n, x)^{-1} \mathbf{W}_h^{n+1} - D(\mathbf{U}^n, x)^{-1} \overline{\mathbf{W}}_h^{n+1}, \overline{\mathbf{v}}_h \right) = \left(\mathcal{C}_h^{n+1} - \overline{\mathcal{C}}_h^{n+1}, \nabla \cdot \overline{\mathbf{v}}_h \right)$$

$$+ \left(\mathcal{C}^{n+1} - \Pi_h^\Gamma \mathcal{C}^{n+1}, \bar{\mathbf{v}}_h \cdot \mathbf{n} \right)_{\partial\Omega} + \sum_{m=1}^M \left(\mathcal{C}^{n+1} - \Pi_h^\Gamma \mathcal{C}^{n+1}, [\bar{\mathbf{v}}_h \cdot \mathbf{n}] \right)_{\Gamma_m}, \quad (4.18)$$

$$\begin{aligned} & \left(\Phi D_\tau (\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}), \bar{\varphi}_h \right) + \left(\nabla \cdot (\mathbf{W}_h^{n+1} - \bar{\mathbf{W}}_h^{n+1}), \bar{\varphi}_h \right) \\ & - \left(D(\mathbf{U}_h^n, x)^{-1} (\mathbf{W}_h^{n+1} - \mathbf{W}^{n+1}) \cdot \mathbf{U}_h^n, \bar{\varphi}_h \right) \\ & - \left((D(\mathbf{U}_h^n, x)^{-1} \mathbf{U}_h^n - D(\mathbf{U}^n, x)^{-1} \mathbf{U}^n) \cdot \mathbf{W}^{n+1}, \bar{\varphi}_h \right) + \left((\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}) q_I^{n+1}, \bar{\varphi}_h \right) = 0. \end{aligned} \quad (4.19)$$

Firstly, we take $\mathbf{v}_h = \mathbf{U}_h^n - Q_h \mathbf{U}^n$ in (4.16) and get

$$\begin{aligned} & \left(\frac{\mu(\mathcal{C}_h^n)}{k(x)} (\mathbf{U}_h^n - Q_h \mathbf{U}^n) + \frac{\mu(\mathcal{C}_h^n)}{k(x)} (Q_h \mathbf{U}^n - \mathbf{U}^n) + \frac{\mu(\mathcal{C}_h^n) - \mu(\mathcal{C}^n)}{k(x)} \mathbf{U}^n, \mathbf{U}_h^n - Q_h \mathbf{U}^n \right) \\ & = \left(P^n - \Pi_h^\Gamma P^n, (\mathbf{U}_h^n - Q_h \mathbf{U}^n) \cdot \mathbf{n} \right)_{\partial\Omega} + \sum_{m=1}^M \left(P^n - \Pi_h^\Gamma P^n, [(\mathbf{U}_h^n - Q_h \mathbf{U}^n) \cdot \mathbf{n}] \right)_{\Gamma_m} \\ & \leq C \|P^n\|_{\bar{H}^3} \|\mathbf{U}_h^n - Q_h \mathbf{U}^n\|_{L^2} h^2 \end{aligned}$$

which implies that

$$\|\mathbf{U}_h^n - \mathbf{U}^n\|_{L^2} \leq C(h^2 + \|\mathcal{C}_h^n - \mathcal{C}^n\|_{L^2}), \quad \text{for } n = 0, 1, \dots, N. \quad (4.20)$$

In particular, we have

$$\|\mathbf{U}_h^0 - L_h \mathbf{U}^0\|_{L^2} \leq C(h^2 + \|\mathcal{C}_h^0 - \mathcal{C}^0\|_{L^2}) \leq Ch^2.$$

and so, by the inverse inequality,

$$\|\mathbf{U}_h^0 - L_h \mathbf{U}^0\|_{L^\infty} \leq Ch^{-d/2} \|\mathbf{U}_h^0 - L_h \mathbf{U}^0\|_{L^2} \leq Ch^{2-d/2}.$$

As a result, there exists a positive constant h_2 such that when $h < h_2$ we have

$$\|\mathbf{U}_h^0\|_{L^\infty} \leq \|\mathbf{U}^0\|_{L^\infty} + 1. \quad (4.21)$$

Secondly, we proceed with a mathematical induction on

$$\|\mathbf{U}_h^n\|_{L^\infty} \leq \|\mathbf{U}^n\|_{L^\infty} + 1, \quad (4.22)$$

which is already proved for $n = 0$. In the following, we assume that it holds for $0 \leq n \leq k$ and try to prove that it also holds for $n = k + 1$.

Taking $\bar{\mathbf{v}}_h = \mathbf{W}_h^{n+1} - \bar{\mathbf{W}}_h^{n+1}$ and $\bar{\varphi}_h = \mathcal{C}_h^{n+1} - \bar{\mathcal{C}}_h^{n+1}$ in (4.18)-(4.19), we obtain

$$\begin{aligned} & D_\tau \left(\frac{1}{2} \|\sqrt{\Phi}(\mathcal{C}_h^{n+1} - \bar{\mathcal{C}}_h^{n+1})\|_{L^2}^2 \right) + \left(D(\mathbf{U}_h^n, x)^{-1} (\mathbf{W}_h^{n+1} - \bar{\mathbf{W}}_h^{n+1}), \mathbf{W}_h^{n+1} - \bar{\mathbf{W}}_h^{n+1} \right) \\ & = - \left((D(\mathbf{U}_h^n, x)^{-1} - D(\mathbf{U}^n, x)^{-1}) \bar{\mathbf{W}}_h^{n+1}, \mathbf{W}_h^{n+1} - \bar{\mathbf{W}}_h^{n+1} \right) \\ & \quad + \left(D(\mathbf{U}_h^n, x)^{-1} \mathbf{U}_h^n \cdot (\mathbf{W}_h^{n+1} - \mathbf{W}^{n+1}), \mathcal{C}_h^{n+1} - \bar{\mathcal{C}}_h^{n+1} \right) \\ & \quad + \left((D(\mathbf{U}_h^n, x)^{-1} \mathbf{U}_h^n - D(\mathbf{U}^n, x)^{-1} \mathbf{U}^n) \cdot \mathbf{W}^{n+1}, \mathcal{C}_h^{n+1} - \bar{\mathcal{C}}_h^{n+1} \right) \end{aligned}$$

$$\begin{aligned}
& - \left((\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}) q_I^{n+1}, \mathcal{C}_h^{n+1} - \bar{\mathcal{C}}_h^{n+1} \right) + \left(\Phi D_\tau(\mathcal{C}^{n+1} - \bar{\mathcal{C}}_h^{n+1}), \mathcal{C}_h^{n+1} - \bar{\mathcal{C}}_h^{n+1} \right) \\
& + \left(\mathcal{C}^{n+1} - \Pi_h^\Gamma \mathcal{C}^{n+1}, \bar{\mathbf{v}}_h \cdot \mathbf{n} \right)_{\partial\Omega} + \sum_{m=1}^M \left(\mathcal{C}^{n+1} - \Pi_h^\Gamma \mathcal{C}^{n+1}, [\bar{\mathbf{v}}_h \cdot \mathbf{n}] \right)_{\Gamma_m} \\
& \leq C \|\bar{\mathbf{W}}_h^{n+1}\|_{L^\infty} \|\mathbf{U}_h^n - \mathbf{U}^n\|_{L^2} \|\mathbf{W}_h^{n+1} - \bar{\mathbf{W}}_h^{n+1}\|_{L^2} \\
& \quad + C \|D(\mathbf{U}_h^n, x)^{-1} \mathbf{U}_h^n\|_{L^\infty} (\|\mathbf{W}_h^{n+1} - \bar{\mathbf{W}}_h^{n+1}\|_{L^2} + \|\mathbf{W}^{n+1} - \bar{\mathbf{W}}_h^{n+1}\|_{L^2}) \|\mathcal{C}_h^{n+1} - \bar{\mathcal{C}}_h^{n+1}\|_{L^2} \\
& \quad + C \|\mathbf{W}^{n+1}\|_{L^\infty} \|\mathbf{U}_h^n - \mathbf{U}^n\|_{L^2} \|\mathcal{C}_h^{n+1} - \bar{\mathcal{C}}_h^{n+1}\|_{L^2} + C \|\mathcal{C}_h^{n+1} - \bar{\mathcal{C}}_h^{n+1}\|_{L^2} \|\mathcal{C}^{n+1} - \bar{\mathcal{C}}_h^{n+1}\|_{L^2} \\
& \quad + C \|D_\tau(\mathcal{C}^{n+1} - \bar{\mathcal{C}}_h^{n+1})\|_{L^2} \|\mathcal{C}_h^{n+1} - \bar{\mathcal{C}}_h^{n+1}\|_{L^2} + C \|\mathcal{C}^{n+1}\|_{\bar{H}^3} \|\mathbf{W}_h^{n+1} - \bar{\mathbf{W}}_h^{n+1}\|_{L^2} h^2 \\
& \leq \frac{1}{2} \left(D(\mathbf{U}_h^n, x)^{-1} (\mathbf{W}_h^{n+1} - \bar{\mathbf{W}}_h^{n+1}), \mathbf{W}_h^{n+1} - \bar{\mathbf{W}}_h^{n+1} \right) \\
& \quad + C(1 + \|\mathbf{U}_h^n\|_{L^\infty}) (\|\mathcal{C}_h^{n+1} - \bar{\mathcal{C}}_h^{n+1}\|_{L^2}^2 + \|\mathbf{U}_h^n - \mathbf{U}^n\|_{L^2}^2 + \|\mathbf{W}^{n+1} - \bar{\mathbf{W}}_h^{n+1}\|_{L^2}^2 + \|\mathcal{C}^{n+1} - \bar{\mathcal{C}}_h^{n+1}\|_{L^2}^2) \\
& \leq \frac{1}{2} \left(D(\mathbf{U}_h^n, x)^{-1} (\mathbf{W}_h^{n+1} - \bar{\mathbf{W}}_h^{n+1}), \mathbf{W}_h^{n+1} - \bar{\mathbf{W}}_h^{n+1} \right) \\
& \quad + C(1 + \|\mathbf{U}_h^n\|_{L^\infty}) (\|\mathcal{C}_h^{n+1} - \bar{\mathcal{C}}_h^{n+1}\|_{L^2}^2 + \|\mathcal{C}_h^n - \bar{\mathcal{C}}_h^n\|_{L^2}^2 + \|D_\tau(\mathcal{C}^{n+1} - \bar{\mathcal{C}}_h^{n+1})\|_{L^2}^2 + h^4).
\end{aligned}$$

Since $\|\mathbf{U}^n\|_{L^\infty} \leq C \|\mathbf{U}^n\|_{\bar{H}^2} \leq C$, by applying (4.22) and Gronwall's inequality, there exists a positive constant τ_4 such that when $\tau < \tau_4$ we have

$$\max_{0 \leq n \leq k} \|\mathcal{C}_h^{n+1} - \bar{\mathcal{C}}_h^{n+1}\|_{L^2}^2 \leq Ch^2,$$

which, together with (4.7), gives

$$\max_{0 \leq n \leq k} \|\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}\|_{L^2} \leq Ch^2. \quad (4.23)$$

From (4.20) we further derive that

$$\max_{0 \leq n \leq k} \|\mathbf{U}_h^{n+1} - \mathbf{U}^{n+1}\|_{L^2} \leq Ch^2, \quad (4.24)$$

which implies that

$$\|\mathbf{U}_h^{k+1} - L_h \mathbf{U}^{k+1}\|_{L^2} \leq Ch^2,$$

and so, by the inverse inequality,

$$\|\mathbf{U}_h^{k+1} - L_h \mathbf{U}^{k+1}\|_{L^\infty} \leq Ch^{-d/2} \|\mathbf{U}_h^{k+1} - L_h \mathbf{U}^{k+1}\|_{L^2} \leq Ch^{2-d/2}.$$

In view of the last inequality, there exists a positive constant h_3 such that when $h < h_3$ we have

$$\|\mathbf{U}_h^{k+1}\|_{L^\infty} \leq \|\mathbf{U}^{k+1}\|_{L^\infty} + 1. \quad (4.25)$$

The mathematical induction on (4.22) is completed, and Proposition 4.1 is proved with $\tau_* = \min(\tau_0, \tau_4)$ and $h_* := \min(h_1, h_2, h_3)$. ■

5 Proof of Theorem 2.1

In this section, we prove Theorem 2.1 based on the boundedness of the fully discrete solution proved in Proposition 4.1.

Similar as the last section, for any fixed integer $n \geq -1$ we introduce the Ritz projection $(\bar{c}_h^{n+1}, \bar{\mathbf{w}}_h^{n+1}) \in S_h^r \times \mathbf{H}_h^r$ of $(c^{n+1}, \mathbf{w}^{n+1}) \in H^1 \times \mathbf{H}_\Gamma^1$ as the finite element solution of

$$\begin{cases} (\nabla \cdot (\bar{\mathbf{w}}_h^{n+1} - \mathbf{w}^{n+1}), \chi_h) = 0, & \forall \chi_h \in S_h^r, \\ (D(\mathbf{u}^{n+1}, x)^{-1}(\bar{\mathbf{w}}_h^{n+1} - \mathbf{w}^{n+1}), \mathbf{v}_h) = (\bar{c}_h^{n+1} - c^{n+1}, \nabla \cdot \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{H}_h^r, \end{cases} \quad (5.1)$$

with $\int_\Omega (\bar{c}_h^{n+1} - c^{n+1}) dx = 0$ for the uniqueness of solution. Then there exists a positive constant $h_{**} \leq h_*$ such that when $h < h_{**}$ the following inequalities hold:

$$\|\phi - \Pi_h \phi\|_{L^2} \leq C \|\phi\|_{\bar{H}^{r+1}} h^{r+1}, \quad \forall \phi \in \bar{H}^{r+1}, \quad (5.2)$$

$$\max_{0 \leq n \leq N} (\|c^n - \bar{c}_h^n\|_{L^2} + \|\mathbf{u}^n - Q_h \mathbf{u}^n\|_{L^2} + \|\mathbf{w}^n - \bar{\mathbf{w}}_h^n\|_{L^2}) \leq C h^{r+1}, \quad (5.3)$$

$$\max_{0 \leq n \leq N} (\|\mathbf{w}^n\|_{L^\infty} + \|\bar{\mathbf{w}}_h^n\|_{L^\infty}) \leq C, \quad (5.4)$$

$$\left(\sum_{n=0}^{N-1} \tau \|D_\tau(c^{n+1} - \bar{c}_h^{n+1})\|_{L^2}^2 \right)^{1/2} \leq C h^{r+1}. \quad (5.5)$$

Firstly, by choosing $\mathbf{v}_h = \mathbf{U}_h^n$ in (2.1) and $\varphi_h = P_h^n$ in (2.2), we derive that

$$\|\mathbf{U}_h^n\|_{L^2}^2 \leq C \|P_h^n\|_{L^2}. \quad (5.6)$$

In order to make use of (5.6), we define \tilde{g}^n as the solution of

$$\begin{cases} \Delta \tilde{g}^n = P_h^n & \text{in } \Omega_m, \\ [\tilde{g}^n] = 0, \quad [\tilde{g}^n \cdot \mathbf{n}] = 0 & \text{on } \Gamma_m, \\ \nabla \tilde{g}^n \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and substitute $\mathbf{v}_h = -Q_h(\nabla \tilde{g}^n)$ into (2.1). Then we obtain

$$\|P_h^n\|_{L^2}^2 = \left(\frac{\mu(\mathcal{C}_h^n)}{k(x)} \mathbf{U}_h^n, \mathbf{v}_h \right) \leq C \|\mathbf{U}_h^n\|_{L^2} \|\mathbf{v}_h\|_{L^2} \leq C \|\mathbf{U}_h^n\|_{L^2} \|P_h^n\|_{L^2},$$

which together with (5.6) implies that

$$\|P_h^n\|_{L^2} + \|\mathbf{U}_h^n\|_{L^2} \leq C. \quad (5.7)$$

When $h \geq h_{**}$, by the inverse inequality we have

$$\|\mathbf{U}_h^n\|_{L^\infty} \leq C h^{-d/2} \|\mathbf{U}_h^n\|_{L^2} \leq C h_{**}^{-d/2} \|\mathbf{U}_h^n\|_{L^2} \leq C. \quad (5.8)$$

Then we choose $\bar{\mathbf{v}}_h = \mathbf{W}_h^{n+1}$ in (2.3) and $\bar{\varphi}_h = \mathcal{C}_h^{n+1}$ in (2.4). With the boundedness of $\|\mathbf{U}_h^n\|_{L^\infty}$, we derive that

$$D_\tau \left(\frac{1}{2} \|\sqrt{\Phi} \mathcal{C}_h^{n+1}\|_{L^2}^2 \right) + \left(D(\mathbf{U}_h^n, x)^{-1} \mathbf{W}_h^{n+1}, \mathbf{W}_h^{n+1} \right)$$

$$\begin{aligned}
&\leq \frac{1}{2} \left(D(\mathbf{U}_h^n, x)^{-1} \mathbf{W}_h^{n+1}, \mathbf{W}_h^{n+1} \right) + C(1 + \|\mathbf{U}_h^n\|_{L^\infty}) \|\mathcal{C}_h^{n+1}\|_{L^2}^2 + C \|\hat{c}^{n+1} q_I^{n+1}\|_{L^2}^2 \\
&\leq \frac{1}{2} \left(D(\mathbf{U}_h^n, x)^{-1} \mathbf{W}_h^{n+1}, \mathbf{W}_h^{n+1} \right) + C \|\mathcal{C}_h^{n+1}\|_{L^2}^2 + C \|\hat{c}^{n+1} q_I^{n+1}\|_{L^2}^2.
\end{aligned}$$

Applying Gronwall's inequality, there exists a positive constant $\tau_5 < \tau_*$ such that when $\tau < \tau_5$ and $h \geq h_{**}$ we have

$$\|\mathcal{C}_h^{n+1}\|_{L^2} \leq C. \quad (5.9)$$

Secondly, we assume that $\tau < \tau_5$ and $h < h_{**}$ so that Proposition 4.1 holds. Note that the mixed formulation of (1.1)-(1.2) gives

$$\begin{aligned}
\left(\frac{\mu(c^n)}{k(x)} \mathbf{u}^n, \mathbf{v}_h \right) &= \left(p^n, \nabla \cdot \mathbf{v}_h \right) - \left(p^n, \mathbf{v}_h \cdot \mathbf{n} \right)_{\partial\Omega} - \sum_{m=1}^M \left(p^n, [\mathbf{v}_h \cdot \mathbf{n}] \right)_{\Gamma_m}, \\
\left(\nabla \cdot \mathbf{u}^n, \varphi_h \right) &= \left(q_I^n - q_P^n, \varphi_h \right), \\
\left(D(\mathbf{u}^{n+1}, x)^{-1} \mathbf{w}^{n+1}, \bar{\mathbf{v}}_h \right) &= \left(c^{n+1}, \nabla \cdot \bar{\mathbf{v}}_h \right) - \left(c^{n+1}, \bar{\mathbf{v}}_h \cdot \mathbf{n} \right)_{\partial\Omega} - \sum_{m=1}^M \left(c^{n+1}, [\bar{\mathbf{v}}_h \cdot \mathbf{n}] \right)_{\Gamma_m}, \\
\left(\Phi(x) D_\tau c^{n+1}, \bar{\varphi}_h \right) &+ \left(\nabla \cdot \mathbf{w}^{n+1}, \bar{\varphi}_h \right) - \left(D(\mathbf{u}^{n+1}, x)^{-1} \mathbf{u}^n \cdot \mathbf{w}^{n+1}, \bar{\varphi}_h \right) \\
&= \left(\hat{c}^{n+1} q_I^{n+1} - c^{n+1} q_I^{n+1}, \bar{\varphi}_h \right) + (\mathcal{E}^{n+1}, \bar{\varphi}_h)
\end{aligned}$$

for any $\mathbf{v}_h, \bar{\mathbf{v}}_h \in \mathbf{H}_h^r$ and $\varphi_h, \bar{\varphi}_h \in S_h$. The above equations with the finite element system (2.1)-(2.4) imply that

$$\begin{aligned}
\left(\frac{\mu(\mathcal{C}_h^n)}{k(x)} \mathbf{U}_h^n - \frac{\mu(c^n)}{k(x)} \mathbf{u}^n, \mathbf{v}_h \right) &= \left(P_h^n - \Pi_h p^n, \nabla \cdot \mathbf{v}_h \right) \\
&+ \left(p^n - \Pi_h^\Gamma p^n, \mathbf{v}_h \cdot \mathbf{n} \right)_{\partial\Omega} + \sum_{m=1}^M \left(p^n - \Pi_h^\Gamma p^n, [\mathbf{v}_h \cdot \mathbf{n}] \right)_{\Gamma_m}, \quad (5.10)
\end{aligned}$$

$$\left(\nabla \cdot (\mathbf{U}_h^n - Q_h \mathbf{u}^n), \varphi_h \right) = 0, \quad (5.11)$$

$$\begin{aligned}
\left(D(\mathbf{U}_h^n, x)^{-1} \mathbf{W}_h^{n+1} - D(\mathbf{u}^{n+1}, x)^{-1} \bar{\mathbf{w}}_h^{n+1}, \bar{\mathbf{v}}_h \right) &= \left(\mathcal{C}_h^{n+1} - \bar{c}_h^{n+1}, \nabla \cdot \bar{\mathbf{v}}_h \right) \\
&+ \left(c^{n+1} - \Pi_h^\Gamma c^{n+1}, \bar{\mathbf{v}}_h \cdot \mathbf{n} \right)_{\partial\Omega} + \sum_{m=1}^M \left(c^{n+1} - \Pi_h^\Gamma c^{n+1}, [\bar{\mathbf{v}}_h \cdot \mathbf{n}] \right)_{\Gamma_m}, \quad (5.12)
\end{aligned}$$

$$\begin{aligned}
&\left(\Phi D_\tau (\mathcal{C}_h^{n+1} - c^{n+1}), \bar{\varphi}_h \right) + \left(\nabla \cdot (\mathbf{W}_h^{n+1} - \bar{\mathbf{w}}_h^{n+1}), \bar{\varphi}_h \right) \\
&- \left(D(\mathbf{U}_h^n, x)^{-1} (\mathbf{W}_h^{n+1} - \mathbf{w}^{n+1}) \cdot \mathbf{U}_h^n, \bar{\varphi}_h \right) \\
&- \left((D(\mathbf{U}_h^n, x)^{-1} \mathbf{U}_h^n - D(\mathbf{u}^{n+1}, x)^{-1} \mathbf{u}^n) \cdot \mathbf{w}^{n+1}, \bar{\varphi}_h \right) + \left((\mathcal{C}_h^{n+1} - c^{n+1}) q_I^{n+1}, \bar{\varphi}_h \right) \\
&= -(\mathcal{E}^{n+1}, \bar{\varphi}_h). \quad (5.13)
\end{aligned}$$

Taking $\mathbf{v}_h = \mathbf{U}_h^n - Q_h \mathbf{u}^n$ in (5.10), we get

$$\begin{aligned} & \left(\frac{\mu(\mathcal{C}_h^n)}{k(x)} (\mathbf{U}_h^n - Q_h \mathbf{u}^n) + \frac{\mu(\mathcal{C}_h^n)}{k(x)} (Q_h \mathbf{u}^n - \mathbf{u}^n) + \frac{\mu(\mathcal{C}_h^n) - \mu(c^n)}{k(x)} \mathbf{u}^n, \mathbf{U}_h^n - Q_h \mathbf{u}^n \right) \\ &= \left(p^n - \Pi_h^\Gamma p^n, (\mathbf{U}_h^n - Q_h \mathbf{u}^n) \cdot \mathbf{n} \right)_{\partial\Omega} + \sum_{m=1}^M \left(p^n - \Pi_h^\Gamma p^n, [(\mathbf{U}_h^n - Q_h \mathbf{u}^n) \cdot \mathbf{n}] \right)_{\Gamma_m}, \\ &\leq C \|p^n\|_{\overline{H}^{r+1}} \|\mathbf{U}_h^n - Q_h \mathbf{u}^n\|_{L^2} h^{r+1}, \end{aligned}$$

which implies that

$$\|\mathbf{U}_h^n - \mathbf{u}^n\|_{L^2} \leq C(h^{r+1} + \|\mathcal{C}_h^n - c^n\|_{L^2}), \quad \text{for } n = 0, 1, \dots, N. \quad (5.14)$$

Taking $\overline{\mathbf{v}}_h = \mathbf{W}_h^{n+1} - \overline{\mathbf{w}}_h^{n+1}$ and $\overline{\varphi}_h = \mathcal{C}_h^{n+1} - \overline{c}_h^{n+1}$ in (5.12)-(5.13), we obtain

$$\begin{aligned} & D_\tau \left(\frac{1}{2} \|\sqrt{\Phi}(\mathcal{C}_h^{n+1} - \overline{c}_h^{n+1})\|_{L^2}^2 \right) + \left(D(\mathbf{U}_h^n, x)^{-1} (\mathbf{W}_h^{n+1} - \overline{\mathbf{w}}_h^{n+1}), \mathbf{W}_h^{n+1} - \overline{\mathbf{w}}_h^{n+1} \right) \\ &= \left((D(\mathbf{U}_h^n, x)^{-1} - D(\mathbf{u}^{n+1}, x)^{-1}) \overline{\mathbf{w}}_h^{n+1}, \mathbf{W}_h^{n+1} - \overline{\mathbf{w}}_h^{n+1} \right) \\ &\quad + \left(D(\mathbf{U}_h^n, x)^{-1} \mathbf{U}_h^n \cdot (\mathbf{W}_h^{n+1} - \mathbf{w}^{n+1}), \mathcal{C}_h^{n+1} - \overline{c}_h^{n+1} \right) \\ &\quad + \left((D(\mathbf{U}_h^n, x)^{-1} \mathbf{U}_h^n - D(\mathbf{u}^{n+1}, x)^{-1} \mathbf{u}^n) \cdot \mathbf{w}^{n+1}, \mathcal{C}_h^{n+1} - \overline{c}_h^{n+1} \right) \\ &\quad - \left((\mathcal{C}_h^{n+1} - c^{n+1}) q_I^{n+1}, \mathcal{C}_h^{n+1} - \overline{c}_h^{n+1} \right) + \left(\Phi D_\tau (c^{n+1} - \overline{c}_h^{n+1}), \mathcal{C}_h^{n+1} - \overline{c}_h^{n+1} \right) \\ &\quad + \left(c^{n+1} - \Pi_h^\Gamma c^{n+1}, \overline{\mathbf{v}}_h \cdot \mathbf{n} \right)_{\partial\Omega} + \sum_{m=1}^M \left(c^{n+1} - \Pi_h^\Gamma c^{n+1}, [\overline{\mathbf{v}}_h \cdot \mathbf{n}] \right)_{\Gamma_m} - \left(\mathcal{E}^{n+1}, \mathcal{C}_h^{n+1} - \overline{c}_h^{n+1} \right) \\ &\leq C \|\overline{\mathbf{w}}_h^{n+1}\|_{L^\infty} \|\mathbf{U}_h^n - \mathbf{u}^{n+1}\|_{L^2} \|\mathbf{W}_h^{n+1} - \overline{\mathbf{w}}_h^{n+1}\|_{L^2} \\ &\quad + C \|D(\mathbf{U}_h^n, x)^{-1} \mathbf{U}_h^n\|_{L^\infty} (\|\mathbf{W}_h^{n+1} - \overline{\mathbf{w}}_h^{n+1}\|_{L^2} + \|\mathbf{w}^{n+1} - \overline{\mathbf{w}}_h^{n+1}\|_{L^2}) \|\mathcal{C}_h^{n+1} - \overline{c}_h^{n+1}\|_{L^2} \\ &\quad + C \|\mathbf{w}^{n+1}\|_{L^\infty} \|\mathbf{U}_h^n - \mathbf{u}^{n+1}\|_{L^2} \|\mathcal{C}_h^{n+1} - \overline{c}_h^{n+1}\|_{L^2} + C \|\mathcal{C}_h^{n+1} - \overline{c}_h^{n+1}\|_{L^2}^2 + C \|\mathcal{E}^{n+1}\|_{L^2}^2 \\ &\leq \frac{1}{2} \left(D(\mathbf{U}_h^n, x)^{-1} (\mathbf{W}_h^{n+1} - \overline{\mathbf{w}}_h^{n+1}), \mathbf{W}_h^{n+1} - \overline{\mathbf{w}}_h^{n+1} \right) + C \|\mathcal{E}^{n+1}\|_{L^2}^2 \\ &\quad + C(1 + \|\mathbf{U}_h^n\|_{L^\infty}^2) (\|\mathcal{C}_h^n - \overline{c}_h^n\|_{L^2}^2 + \|\mathcal{C}_h^{n+1} - \overline{c}_h^{n+1}\|_{L^2}^2 + \|D_\tau (c^{n+1} - \overline{c}_h^{n+1})\|_{L^2}^2 + h^{2r+2} + \tau^2). \end{aligned}$$

Applying Gronwall's inequality, there exists a positive constant $\tau_{**} < \tau_5$ such that, when $\tau < \tau_{**}$ and $h < h_{**}$, Proposition 4.1 holds and the last inequality reduces to

$$\max_{1 \leq n \leq N} \|\mathcal{C}_h^n - \overline{c}_h^n\|_{L^2}^2 + \sum_{n=0}^{N-1} \tau \|\mathbf{W}_h^{n+1} - \overline{\mathbf{w}}_h^{n+1}\|_{L^2}^2 \leq C(\tau + h^{r+1})^2,$$

which together with (5.14) implies that

$$\max_{1 \leq n \leq N} (\|\mathcal{C}_h^n - c^n\|_{L^2} + \|\mathbf{U}_h^n - \mathbf{u}^n\|_{L^2}) + \left(\sum_{n=1}^N \tau \|\mathbf{W}_h^n - \mathbf{w}^n\|_{L^2}^2 \right)^{\frac{1}{2}} \leq C(\tau + h^{r+1}). \quad (5.15)$$

To estimate $\|P_h^n - p^n\|_{L^2}$, we define g^n as the solution of

$$\begin{cases} \Delta g^n = P_h^n - \Pi_h p^n & \text{in } \Omega_m, \\ [g^n] = 0, \quad [\nabla g^n \cdot \mathbf{n}] = 0 & \text{on } \Gamma_m, \\ \nabla g^n \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and substitute $\mathbf{v}_h = -Q_h(\nabla g^n)$ into (5.10). Since $\|\mathbf{v}_h\|_{L^2} \leq C\|P_h^n - p^n\|_{L^2}$, it follows that

$$\begin{aligned} \|P_h^n - \Pi_h p^n\|_{L^2}^2 &\leq C(\|\mathcal{C}_h^n - c^n\|_{L^2} + \|\mathbf{U}_h^n - \mathbf{u}^n\|_{L^2} + C\|p^n\|_{\overline{H}^{r+1}} h^{r+1}) \|\mathbf{v}_h\|_{L^2} \\ &\leq C(\|\mathcal{C}_h^n - c^n\|_{L^2} + \|\mathbf{U}_h^n - \mathbf{u}^n\|_{L^2} + C\|p^n\|_{\overline{H}^{r+1}} h^{r+1}) \|P_h^n - \Pi_h p^n\|_{L^2} \\ &\leq C(\tau + h^{r+1}) \|P_h^n - \Pi_h p^n\|_{L^2}, \end{aligned}$$

which gives

$$\max_{1 \leq n \leq N} \|P_h^n - \Pi_h p^n\|_{L^2} \leq C(\tau + h^{r+1}). \quad (5.16)$$

Finally, when $\tau < \tau_{**}$ and $h \geq h_{**}$ we see that (5.7)-(5.9) give

$$\begin{aligned} \max_{1 \leq n \leq N} (\|P_h^n - p^n\|_{L^2} + \|\mathbf{U}_h^n - \mathbf{u}^n\|_{L^2} + \|\mathcal{C}_h^n - c^n\|_{L^2}) &+ \left(\sum_{n=1}^N \tau \|\mathbf{W}_h^n - \mathbf{w}^n\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\leq C \leq \frac{C}{h^{r+1}} (\tau + h^{r+1}). \end{aligned} \quad (5.17)$$

From (5.15)-(5.17) we see that Theorem 2.1 holds. ■

6 Proof of Lemma 2.1 and Lemma 2.2

In this section, we prove Lemma 2.1 and Lemma 2.2, which were used in Section 3 to prove the uniform piecewise regularity of the solution of the linearized PDEs. We shall use the notation $x = (x', x_d)$, with $x' = (x_1, \dots, x_{d-1})$.

6.1 Proof of Lemma 2.1

Before we prove Lemma 2.1, we need to introduce some lemmas below.

Lemma 6.1 *Let $S_R = \{x \in \mathbb{R}^d : |x'| < R \text{ and } |x_d| < R\}$, $S_R^+ = S_R \cap \{x \in \mathbb{R}^d : 0 < x_d < R\}$, $S_R^- = S_R \cap \{x \in \mathbb{R}^d : -R < x_d < 0\}$. Let $\Gamma = \{x \in \mathbb{R}^d : x_d = 0\}$ and $\Gamma_R := S_R \cap \Gamma$. Suppose that $A_{ij} = A_{ji} \in H^2(S_{2R}^+) \cap H^2(S_{2R}^-)$ satisfies the strong ellipticity condition*

$$K^{-1}|\xi|^2 \leq \sum_{i,j=1}^d A_{ij}(x) \xi_i \xi_j \leq K|\xi|^2 \quad \text{for } x \in S_{2R} \setminus \Gamma \text{ and } \xi \in \mathbb{R}^d,$$

and $\phi \in H^3(S_{2R}^+) \cap H^3(S_{2R}^-)$ is a solution of

$$\begin{cases} -\nabla \cdot (A \nabla \phi) = f & \text{in } S_{2R} \setminus \Gamma_{2R}, \\ [\phi] = 0, \quad [A \nabla \phi \cdot \mathbf{n}] = g & \text{on } \Gamma_{2R}. \end{cases} \quad (6.1)$$

Then

$$\|\phi\|_{\overline{H}^2(S_R)} \leq C_R (\|f\|_{L^2(S_{3R/2})} + \|g\|_{H^{1/2}(\Gamma_{3R/2})} + \|\phi\|_{\overline{W}^{1,3}(S_{3R/2})}), \quad (6.2)$$

$$\|\phi\|_{\overline{H}^3(S_R)} \leq C_R (\|f\|_{\overline{H}^1(S_{2R})} + \|g\|_{H^{3/2}(\Gamma_{2R})} + \|\phi\|_{\overline{W}^{2,4}(S_{2R})}). \quad (6.3)$$

where $\|\psi\|_{\overline{H}^k(S_R)} := \|\psi\|_{H^k(S_R^+)} + \|\psi\|_{H^k(S_R^-)}$ for any $\psi \in H^k(S_R^+) \cap H^k(S_R^-)$ and nonnegative integer k .

Proof To simplify the notations, we relax the dependence on R in the generic constant, and set $(f_1, f_2) = \int_{\Omega} f_1(x)f_2(x) dx$, $(g_1, g_2)_{\Gamma} = \int_{\Gamma} g_1(x')g_2(x') dx'$.

Differentiating the equation (6.1) with respect to x_j for some fixed $1 \leq j \leq d-1$ and denote $\phi_j = \partial_j \phi$, we obtain that

$$\begin{cases} -\nabla \cdot (A \nabla \phi_j) = \partial_j f + \nabla \cdot (\partial_j A \nabla \phi) & \text{in } S_{2R} \setminus \Gamma_{2R}, \\ [\phi_j] = 0, \quad [A \nabla \phi_j \cdot \mathbf{n}] = \partial_j g - [\partial_j A \nabla \phi \cdot \mathbf{n}] & \text{on } \Gamma_{2R}. \end{cases} \quad (6.4)$$

where \mathbf{n} denote the upward unit normal vector on Γ .

If we define ζ_R as a smooth cut-off function satisfying $0 \leq \zeta_R \leq 1$, $\zeta_R = 1$ in S_R and $\zeta_R = 0$ outside $S_{3R/2}$, then (6.4) times $\phi_j \zeta_R^2$ gives

$$\begin{aligned} & (\zeta_R^2 A \nabla \phi_j, \nabla \phi_j) + (2\zeta_R \phi_j A \nabla \phi_j, \nabla \zeta_R) \\ &= -(f \zeta_R^2, \partial_j \phi_j) - (2\phi_j f \zeta_R, \partial_j \zeta_R) - (\zeta_R^2 \partial_j A \nabla \phi, \nabla \phi_j) - (2\zeta_R \phi_j \partial_j A \nabla \phi, \nabla \zeta_R) + (\partial_j g, \phi_j \zeta_R^2)_{\Gamma}, \end{aligned}$$

which reduces to

$$\begin{aligned} \|\nabla \phi_j\|_{L^2(S_R)}^2 &\leq C(\|\phi_j\|_{L^2(S_{3R/2})}^2 + \|f\|_{L^2(S_{3R/2})}^2 + \|\partial_j A\|_{L^6(S_{2R})}^2 \|\nabla \phi\|_{L^3(S_{3R/2})}^2 \\ &\quad + \|\partial_j A\|_{L^6(S_{2R})} \|\nabla \phi\|_{L^{12/5}(S_{3R/2})}^2 + \|g \partial_j \zeta_R\|_{L^2(\Gamma)} \|\phi_j \zeta_R\|_{L^2(\Gamma)} \\ &\quad + \|\partial_j (g \zeta_R)\|_{H^{-1/2}(\Gamma)} \|\phi_j \zeta_R\|_{H^{1/2}(\Gamma)}) \\ &\leq C(\|f\|_{L^2(S_{3R/2})}^2 + \|g\|_{H^{1/2}(\Gamma_{3R/2})}^2 + \|\phi_j\|_{H^1(S_{3R/2})}^2 + \|\nabla \phi\|_{L^3(S_{3R/2})}^2), \end{aligned}$$

and from (6.1) we see that

$$\begin{aligned} \|\partial_{dd} \phi\|_{\overline{L}^2(S_R)} &= \left\| A_{dd}^{-1} \left(\sum_{(i,j) \neq (d,d)} A_{ij} \partial_{ij} \phi + \sum_{i,j=1}^d \partial_i A_{ij} \partial_j \phi + f \right) \right\|_{\overline{L}^2(S_R)} \\ &\leq C(\|f\|_{L^2(S_{3R/2})} + \|g\|_{H^{1/2}(\Gamma_{3R/2})} + \|\phi_j\|_{H^1(S_{3R/2})} + \|\nabla \phi\|_{L^3(S_{3R/2})}). \end{aligned}$$

The last two inequalities imply (6.2).

By applying (6.2) to the problem (6.4), we derive that

$$\begin{aligned} \|\phi_j\|_{\overline{H}^2(S_R)} &\leq C(\|\partial_j f\|_{\overline{L}^2(S_{3R/2})} + \|\nabla \cdot (\partial_j A \nabla \phi)\|_{\overline{L}^2(S_{3R/2})} + \|\partial_j g\|_{H^{1/2}(\Gamma_{3R/2})} \\ &\quad + \|[\partial_j A \nabla \phi \cdot \mathbf{n}]\|_{H^{1/2}(\Gamma_{3R/2})} + \|\phi_j\|_{\overline{W}^{1,3}(S_{3R/2})}) \\ &\leq C(\|f\|_{\overline{H}^1(S_{2R})} + \|A\|_{\overline{H}^2(S_{2R})} \|\phi\|_{\overline{W}^{1,\infty}(S_{2R})} + \|A\|_{\overline{W}^{1,6}(S_{2R})} \|\phi\|_{\overline{W}^{2,3}(S_{2R})} \\ &\quad + \|g\|_{H^{3/2}(\Gamma_{2R})} + \|\partial_j A \nabla \phi \cdot \mathbf{n}\|_{\overline{H}^1(S_{2R})} + \|\phi_j\|_{\overline{W}^{1,3}(S_{2R})}) \\ &\leq C(\|f\|_{\overline{H}^1(S_{2R})} + \|g\|_{H^{3/2}(\Gamma_{2R})} + \|\phi\|_{\overline{W}^{2,3}(S_{2R})} \\ &\quad + \|A\|_{\overline{H}^2(S_{2R})} \|\phi\|_{\overline{W}^{1,\infty}(S_{2R})} + \|A\|_{\overline{W}^{1,6}(S_{2R})} \|\phi\|_{\overline{W}^{2,3}(S_{2R})}) \\ &\leq C(\|f\|_{\overline{H}^1(S_{2R})} + \|g\|_{H^{3/2}(\Gamma_{2R})} + \|\phi\|_{\overline{W}^{2,4}(S_{2R})}). \end{aligned}$$

Then from (6.1) we derive that

$$\|\partial_{dd} \phi_d\|_{\overline{L}^2(S_R)} = \left\| A_{dd}^{-1} \left(\sum_{(l,k) \neq (d,d)} A_{lk} \partial_{lk} \phi_d + \sum_{l,k=1}^d \partial_l A_{lk} \partial_k \phi_d + \partial_d f + \nabla \cdot (\partial_d A \nabla \phi) \right) \right\|_{\overline{L}^2(S_R)}$$

$$\leq C(\|f\|_{\overline{H}^1(S_{2R})} + \|g\|_{H^{3/2}(\Gamma_{2R})} + \|\phi\|_{\overline{W}^{2,4}(S_{2R})}).$$

The last two inequalities imply (6.3), and the proof of Lemma 6.1 is completed. ■

The above lemma can be easily extended to the case that $\Gamma(\varphi)$ is a smooth surface defined by $x_d = \varphi(x')$ for some smooth function $\varphi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$.

Lemma 6.2 *Let $S_R(\varphi) = \{x \in \mathbb{R}^d : |x'| < R \text{ and } \varphi(x') - R < x_d < \varphi(x') + R\}$, $S_R^+(\varphi) = \{x \in \mathbb{R}^d : |x'| < R \text{ and } \varphi(x') < x_d < \varphi(x') + R\}$, $S_R^- = \{x \in \mathbb{R}^d : |x'| < R \text{ and } \varphi(x') - R < x_d < \varphi(x')\}$, and $\Gamma_R(\varphi) = S_R(\varphi) \cap \{x_d = \varphi(x')\}$. Suppose that $A_{ij} = A_{ji} \in H^3(S_{2R}^+(\varphi)) \cap H^3(S_{2R}^-(\varphi))$ satisfies that*

$$K^{-1}|\xi|^2 \leq \sum_{i,j=1}^d A_{ij}(x)\xi_i\xi_j \leq K|\xi|^2 \quad \text{for } x \in S_{2R}(\varphi) \setminus \Gamma_{2R}(\varphi) \text{ and } \xi \in \mathbb{R}^d,$$

and assume that $\phi \in H^3(S_{2R}^+(\varphi)) \cap H^3(S_{2R}^-(\varphi))$ is a solution of

$$\begin{cases} -\nabla \cdot (A \nabla \phi) = f & \text{in } S_{2R}(\varphi) \setminus \Gamma_{2R}(\varphi), \\ [\phi] = 0, \quad [A \nabla \phi \cdot \mathbf{n}] = g & \text{on } \Gamma_{2R}(\varphi). \end{cases} \quad (6.5)$$

Then

$$\|\phi\|_{\overline{H}^2(S_R(\varphi))} \leq C_R(\|f\|_{L^2(S_{2R}(\varphi))} + \|g\|_{H^{1/2}(\Gamma_{2R}(\varphi))} + \|\phi\|_{\overline{W}^{1,3}(S_{2R}(\varphi))}), \quad (6.6)$$

$$\|\phi\|_{\overline{H}^3(S_R(\varphi))} \leq C_R(\|f\|_{\overline{H}^1(S_{2R}(\varphi))} + \|g\|_{H^{3/2}(\Gamma_{2R}(\varphi))} + \|\phi\|_{\overline{W}^{2,4}(S_{2R}(\varphi))}), \quad (6.7)$$

where $\|\psi\|_{\overline{H}^k(S_R(\varphi))} = \|\psi\|_{H^k(S_R^+(\varphi))} + \|\psi\|_{H^k(S_R^-(\varphi))}$ for any $\psi \in H^k(S_R^+(\varphi)) \cup H^k(S_R^-(\varphi))$ and nonnegative integer k .

Proof Let $x = \Psi(y)$ denote the coordinates transformation $x' = y'$ and $x_d = y_d + \varphi(y')$. Under this coordinates transformation, the problem (6.5) is converted to

$$\begin{cases} -\nabla_y \cdot (\tilde{A}(y) \nabla_y \tilde{\phi}) = \tilde{f}(y) & \text{in } S_{2R} \setminus \Gamma_{2R}, \\ [\tilde{\phi}] = 0, \quad [\tilde{A}(y) \nabla_y \tilde{\phi} \cdot \mathbf{n}] = \tilde{g}(y) & \text{on } \Gamma_{2R}, \end{cases} \quad (6.8)$$

where $\tilde{\phi}(y) = \phi(\Psi(y))$, $\tilde{A}(y) = A(\Psi(y))$, $\tilde{f}(y) = f(\Psi(y))$ and $\tilde{g}(y) = g(\Psi(y))\sqrt{1 + |\nabla \varphi(y')|^2}$. By applying Lemma 6.1 to the problem (6.8), we get

$$\begin{aligned} & \|\tilde{\phi}\|_{H^2(S_R^+)} + \|\tilde{\phi}\|_{H^2(S_R^-)} \\ & \leq C_R(\|\tilde{f}\|_{L^2(S_{2R})} + \|\tilde{g}\|_{H^{1/2}(\Gamma_{2R})} + \|\tilde{\phi}\|_{W^{1,3}(S_{2R}^+)} + \|\tilde{\phi}\|_{W^{1,3}(S_{2R}^-)}), \\ & \|\tilde{\phi}\|_{H^3(S_R^+)} + \|\tilde{\phi}\|_{H^3(S_R^-)} \\ & \leq C_R(\|\tilde{f}\|_{H^1(S_{2R}^+)} + \|\tilde{f}\|_{H^1(S_{2R}^-)} + \|\tilde{g}\|_{H^{3/2}(\Gamma_{2R})} + \|\tilde{\phi}\|_{W^{2,4}(S_{2R}^+)} + \|\tilde{\phi}\|_{W^{2,4}(S_{2R}^-)}). \end{aligned}$$

Transforming back to the x -coordinates, the last two inequalities imply (6.6)-(6.7). ■

Proof of Lemma 2.1 Without loss of generality, we can assume that the functions A_{ij} , f and g are sufficiently smooth so that the problem (2.11) has a piecewise H^3 solution [4]. If we can prove (2.12) with a constant C_R which does not depend on the extra smoothness of A_{ij} , f and g , then a compactness argument gives (2.12) for the nonsmooth A_{ij} , f and g under the condition of Lemma 2.1.

First, multiplying the equation (2.11) by ϕ , it is easy to derive the basic H^1 estimate:

$$\|\phi\|_{H^1(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \sum_{m=1}^M \|g\|_{H^{-1/2}(\Gamma_m)} \right).$$

Secondly, by a “partition of unity”, there exist a finite number of cylinders $S_{2R,j} \subset \Omega$, $j = 1, \dots, J$, such that $\{S_{R,j}\}_{j=1}^J$ covers Γ_m , $m = 1, \dots, M$. Moreover, each $S_{2R,j}$ only intersects one interface Γ_m and in each $S_{2R,j}$, up to a rotation, the interface Γ_m can be expressed as $x_d = \varphi_j(x')$ for some smooth function $\varphi_j : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$. Then, by applying Lemma 6.2, we derive that

$$\begin{aligned} \|\phi\|_{\overline{H}^2(S_{R,j})} &\leq C_R (\|f\|_{L^2(S_{2R,j})} + \|g\|_{H^{1/2}(\Gamma_m \cap S_{2R,j})} + \|\phi\|_{\overline{W}^{1,3}(S_{2R,j})}), \quad j = 1, \dots, J, \\ \|\phi\|_{\overline{H}^3(S_{R,j})} &\leq C_R (\|f\|_{\overline{H}^1(S_{2R,j})} + \|g\|_{H^{3/2}(\Gamma_m \cap S_{2R,j})} + \|\phi\|_{\overline{W}^{2,4}(S_{2R,j})}), \quad j = 1, \dots, J. \end{aligned}$$

Let $D = \Omega \setminus \cup_{j=1}^J S_{R,j}$. It is well-known that, by the interior estimates of elliptic equations, there hold

$$\begin{aligned} \|\phi\|_{H^2(D)} &\leq C_R (\|f\|_{L^2(\Omega)} + \|\phi\|_{H^1(\Omega)}), \\ \|\phi\|_{H^3(D)} &\leq C_R (\|f\|_{\overline{H}^1(\Omega)} + \|\phi\|_{H^1(\Omega)}). \end{aligned}$$

The last four inequalities imply that

$$\begin{aligned} \|\phi\|_{\overline{H}^2(\Omega)} &\leq C_R \left(\|f\|_{L^2(\Omega)} + \sum_{m=1}^M \|g\|_{H^{1/2}(\Gamma_m)} + \|\phi\|_{\overline{W}^{1,3}(\Omega)} \right) \\ &\leq C_R \left(\|f\|_{L^2(\Omega)} + \sum_{m=1}^M \|g\|_{H^{1/2}(\Gamma_m)} + C_\epsilon \|\phi\|_{H^1(\Omega)} + \epsilon \|\phi\|_{\overline{H}^2(\Omega)} \right), \\ \|\phi\|_{\overline{H}^3(\Omega)} &\leq C_R (\|f\|_{\overline{H}^1(\Omega)} + \sum_{m=1}^M \|g\|_{H^{3/2}(\Gamma_m)} + \|\phi\|_{\overline{W}^{2,4}(\Omega)}) \\ &\leq C_R \left(\|f\|_{\overline{H}^1(\Omega)} + \sum_{m=1}^M \|g\|_{H^{3/2}(\Gamma_m)} + C_\epsilon \|\phi\|_{H^1(\Omega)} + \epsilon \|\phi\|_{\overline{H}^3(\Omega)} \right), \end{aligned}$$

where ϵ can be arbitrarily small.

Finally, by choosing ϵ small enough and using the basic H^1 estimate, the last two inequalities imply (2.12). ■

6.2 Proof of Lemma 2.2

Integrating (2.13) against ϕ^{n+1} , it is easy to derive that

$$\max_{0 \leq n \leq N-1} \|\phi^{n+1}\|_{L^2}^2 + \sum_{n=0}^{N-1} \tau \|\nabla \phi^{n+1}\|_{L^2}^2 \leq C \sum_{n=0}^{N-1} \tau (\|f^{n+1}\|_{L^2}^2 + \|\mathbf{g}^{n+1}\|_{L^2}^2).$$

Then, by setting $\mathbf{g}^0 = \mathbf{g}^1$, integrating (2.13) against $-\frac{1}{\Phi}\nabla \cdot (A^{n+1}\nabla\phi^{n+1} - \mathbf{g}^{n+1})$ gives

$$\begin{aligned} & (D_\tau\nabla\phi^{n+1}, A^{n+1}\nabla\phi^{n+1} - \mathbf{g}^{n+1}) + \frac{1}{2}\|\Phi^{-1/2}\nabla \cdot (A^{n+1}\nabla\phi^{n+1})\|_{L^2}^2 \\ & \leq C\|f^{n+1}\|_{L^2}^2 + C\|\nabla \cdot \mathbf{g}^{n+1}\|_{L^2}^2 \end{aligned}$$

which further reduces to

$$\begin{aligned} & D_\tau[(A^{n+1}\nabla\phi^{n+1}, \nabla\phi^{n+1}) - 2(\nabla\phi^{n+1}, \mathbf{g}^{n+1})] + \|\Phi^{-1/2}\nabla \cdot (A^{n+1}\nabla\phi^{n+1})\|_{L^2}^2 \\ & \leq C\|f^{n+1}\|_{L^2}^2 + C\|\nabla \cdot \mathbf{g}^{n+1}\|_{L^2}^2 + C(D_\tau A^{n+1}\nabla\phi^n \cdot \nabla\phi^n) - (\nabla\phi^n, D_\tau\mathbf{g}^{n+1}) \\ & \leq C\|f^{n+1}\|_{L^2}^2 + C\|\mathbf{g}^{n+1}\|_{H^1}^2 + C\|D_\tau A^{n+1}\|_{L^2}\|\nabla\phi^n\|_{L^4}^2 + \|\nabla\phi^n\|_{L^2}\|D_\tau\mathbf{g}^{n+1}\|_{L^2}d_{n,0} \\ & \leq C\|f^{n+1}\|_{L^2}^2 + C\|\mathbf{g}^{n+1}\|_{H^1}^2 + C_\epsilon\|\nabla\phi^n\|_{L^2}^2 + \epsilon(\|\nabla\phi^n\|_{H^1}^2 + \|D_\tau\mathbf{g}^{n+1}\|_{L^2}^2d_{n,0}). \end{aligned}$$

From Lemma 2.1 we know that

$$\begin{aligned} \|\phi\|_{H^2} & \leq C\|\Phi^{-1/2}\nabla \cdot (A^{n+1}\nabla\phi^{n+1})\|_{L^2} + C\|\mathbf{g}^{n+1} \cdot \mathbf{n}\|_{H^{1/2}(\Gamma)} \\ & \leq C\|\Phi^{-1/2}\nabla \cdot (A^{n+1}\nabla\phi^{n+1})\|_{L^2} + C\|\mathbf{g}^{n+1}\|_{H^1} \end{aligned}$$

The last two inequalities imply that

$$\begin{aligned} & D_\tau[(A^{n+1}\nabla\phi^{n+1}, \nabla\phi^{n+1}) - 2(\nabla\phi^{n+1}, \mathbf{g}^{n+1})] + C^{-1}\|\phi\|_{H^2}^2 \\ & \leq C\|f^{n+1}\|_{L^2}^2 + C\|\mathbf{g}^{n+1}\|_{H^1}^2 + C_\epsilon\|\nabla\phi^n\|_{L^2}^2 + \epsilon(\|\nabla\phi^n\|_{H^1}^2 + \|D_\tau\mathbf{g}^{n+1}\|_{L^2}^2d_{n,0}). \end{aligned}$$

Summing up the last inequality for $n = 0, 1, \dots, m$, we obtain

$$\begin{aligned} \max_{0 \leq n \leq m} \|\nabla\phi^{n+1}\|_{L^2}^2 + \sum_{n=0}^m \tau\|\phi^{n+1}\|_{H^2}^2 & \leq C_\epsilon \sum_{n=0}^m \tau(\|f^{n+1}\|_{L^2}^2 + \|\mathbf{g}^{n+1}\|_{H^1}^2 + \|\nabla\phi^n\|_{L^2}^2) \\ & \quad + \epsilon \sum_{n=0}^m \tau(\|\nabla\phi^n\|_{H^1}^2 + \|D_\tau\mathbf{g}^{n+1}\|_{L^2}^2d_{n,0}) + C\|\mathbf{g}^{m+1}\|_{L^2}^2, \end{aligned}$$

which further reduces to (2.14). The proof of Lemma 2.2 is completed. \blacksquare

7 Numerical examples

In this section, we present numerical examples to support our theoretical error analysis. The computations are performed with the software FreeFEM++ [21].

We solve the problem

$$\begin{cases} \Phi(x)\frac{\partial c}{\partial t} - \nabla \cdot (D(\mathbf{u}, x)\nabla c) + \mathbf{u} \cdot \nabla c = f & \text{in } \Omega_0 \cup \Omega_1, \\ [c] = 0, \quad [D(\mathbf{u}, x)\nabla c \cdot \mathbf{n}] = 0 & \text{on } \Gamma, \\ D(\mathbf{u}, x)\nabla c \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \\ c(x, 0) = c_0(x) & \text{for } x \in \Omega, \end{cases} \quad (7.1)$$

$$\begin{cases} \nabla \cdot \mathbf{u} = g & \text{in } \Omega_0 \cup \Omega_1, \\ \mathbf{u} = -\frac{k(x)}{\mu(c)} \nabla p & \text{in } \Omega_0 \cup \Omega_1, \\ [p] = 0, \quad [\mathbf{u} \cdot \mathbf{n}] = 0 & \text{on } \Gamma, \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (7.2)$$

in the unit ball $\Omega = \{(x_1, x_2) : |x_1|^2 + |x_2|^2 < 1\}$ which is separated by the interface

$$\Gamma_1 = \{(x_1, x_2) : |x_1 - 0.3|^2 + |x_2|^2 = 0.3^2\}$$

into two subdomains

$$\Omega_0 = \{(x_1, x_2) : |x_1 - 0.3|^2 + |x_2|^2 > 0.3^2\} \quad \text{and} \quad \Omega_1 = \{(x_1, x_2) : |x_1 - 0.3|^2 + |x_2|^2 < 0.3^2\}.$$

For simplicity, we choose $\mu(c) = 1/(1 + e^{5c})$, $d_0 = d_r = d_p = \alpha_1 = \alpha_2 = 1.0$ and choose the permeability and porosity

$$\Phi(x) = \begin{cases} 0.6 & \text{for } x \in \Omega_0, \\ 0.4 & \text{for } x \in \Omega_1, \end{cases} \quad k(x) = \begin{cases} 0.012 & \text{for } x \in \Omega_0, \\ 0.008 & \text{for } x \in \Omega_1, \end{cases}$$

which are smooth in each subdomain but discontinuous across the interface Γ_1 .

Let $\Omega_K = \{(x_1, x_2) : \sqrt{(x_1 - 0.3)^2 + x_2^2} < 0.6\}$ so that $\Omega_1 \subset \Omega_K \subset \Omega$. The functions f , g and the initial data c_0 are chosen corresponding to the exact solution

$$p(x, t) = \begin{cases} 100((x_1 - 0.3)^2 + x_2^2 - 0.09)(0.36 - (x_1 - 0.3)^2 - x_2^2)^4 / \Phi(x), & \text{for } x \in \Omega_K, \\ 0 & \text{for } x \in \Omega \setminus \Omega_K, \end{cases}$$

$$c(x, t) = 0.5 + 50 p(x, t) \cos(0.4 x_1) \sin(0.4 x_2) \sin(4t),$$

which satisfy the jump conditions on the interface Γ_1 and the boundary conditions on $\partial\Omega$, while ∇p and ∇c are discontinuous across the interface Γ_1 .

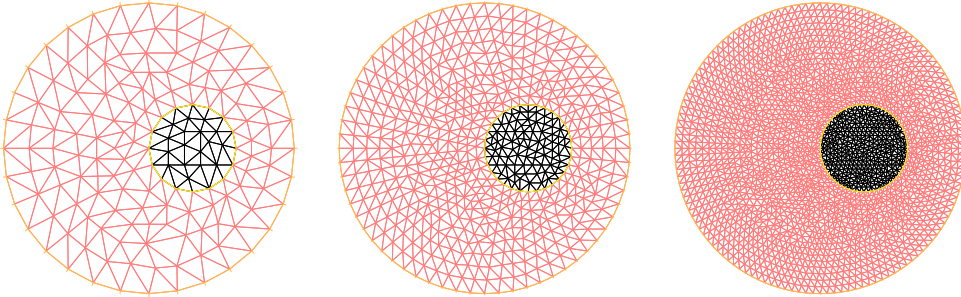


Figure 7.2: The finite element meshes with $h = 1/32$, $h = 1/64$ and $h = 1/128$.

We partition the domain Ω into quasi-uniform triangles with M nodes on the boundary $\partial\Omega$ and $M/2$ nodes on the interface Γ with $M = 32, 64, 128$, as shown in Figure 7.2. For simplicity, we set $h = 1/M$ and solve the system by the proposed method using the Raviart–Thomas linear finite elements up to the time $t = 1$. To test the convergence rate of the proposed method, we

solve the problem for different τ and h , and present the errors of the numerical solutions in Table 7.1, where the convergence rates of \mathbf{U}_h^N and \mathcal{C}_h^N are calculated by the formulas

$$\begin{aligned} \text{convergence rate of } \mathbf{U}_h &= \ln (\|\mathbf{U}_h - \mathbf{u}\|_{L^2} / \|\mathbf{U}_{h/2} - \mathbf{u}\|_{L^2}) / \ln 2, \\ \text{convergence rate of } \mathcal{C}_h &= \ln (\|\mathcal{C}_h - c\|_{L^2} / \|\mathcal{C}_{h/2} - c\|_{L^2}) / \ln 2, \end{aligned}$$

at the finest two meshes. From Table 7.1 we see that the convergence rate of the numerical solution is about second order, which is consistent with our numerical analysis.

Table 7.1: Errors of the linearized mixed FEM with $\tau = O(h^2)$.

τ	h	$\ \mathbf{U}_h^N - \mathbf{u}^N\ _{L^2}$	$\ \mathcal{C}_h^N - c^N\ _{L^2}$
1/8	1/32	3.051E-02	1.473E-02
1/32	1/64	9.769E-03	4.280E-03
1/128	1/128	2.515E-03	1.020E-03
convergence rate		1.96	2.06

To illustrate the convergence rate with respect to τ , we solve the system for fixed τ and several different h . The errors of the numerical solution are present in Table 7.2, where we can see that the error tends to a constant proportional to τ (as h decreases).

Table 7.2: Errors of the linearized mixed FEM with fixed τ and refined h .

$\tau = 0.2$	h	$\ U_h^N - \mathbf{u}(\cdot, t_N)\ _{L^2}$	$\ \mathcal{C}_h^N - c(\cdot, t_N)\ _{L^2}$
	1/32	3.469E-02	2.955E-02
	1/64	2.942E-02	3.031E-02
	1/96	2.887E-02	3.025E-02
	1/128	2.877E-02	3.018E-02
$\tau = 0.1$	h	$\ U_h^N - \mathbf{u}(\cdot, t_N)\ _{L^2}$	$\ \mathcal{C}_h^N - c(\cdot, t_N)\ _{L^2}$
	1/32	2.840E-02	1.438E-02
	1/64	2.008E-02	1.328E-02
	1/96	1.904E-02	1.315E-02
	1/128	1.883E-02	1.308E-02
$\tau = 0.05$	h	$\ U_h^N - \mathbf{u}(\cdot, t_N)\ _{L^2}$	$\ \mathcal{C}_h^N - c(\cdot, t_N)\ _{L^2}$
	1/32	2.429E-02	1.033E-02
	1/64	1.230E-02	6.414E-03
	1/96	1.029E-02	6.100E-03
	1/128	9.848E-03	6.010E-03

8 Conclusions

We have studied the convergence of a linearized mixed FEM for a nonlinear elliptic-parabolic interface problem from the model of incompressible miscible flow in porous media. We showed that the solution of the linearized PDEs is piecewise uniformly regular in each subdomain

separated by the interfaces if the solution of the original problem is piecewise regular, and established optimal-order error estimates for the fully discrete solution without restriction on the grid ratio. The analysis presented in this paper, together with Lemma 2.1–2.2, may be extended to other nonlinear parabolic interface problems with other time-stepping schemes.

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